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DEPARTMENT OF CIVIL ENGINEERING

RADIATION FROM A SOUND SOURCE INSIDE A CYLINDRICAL SHELL SUBMERGED IN AN INFINITE MEDIUM

By

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Office of Naval Research
Contract Nonr-266(08)
Technical Report No.9
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List of Symbols*

θ, z, r	Cylindrical coordinates, Fig. 1
u, v, w	Longitudinal, tangential and radial displacements of the cylindrical shell. Fig. 1. Note that a positive displacement w is inward.
U_k, V_k, W_k	Coefficients defining the shape of modes of the shell in vacuo, Eq. (1)
$k = 1, 2, 3$	Subscript
a	Radius of cylinder
c	Velocity of sound in medium
c_s	Velocity of sound in shell material
D	Subscript, refers to damping
$F(r)$	r-dependent function in expression for potential, Eq. (8)
$g(z)$	Distribution of force $\overline{P}$ in z-direction
$G(\xi)$	Fourier Inverse of $g(z)$
$H_n^{(2)}$	Hankel Function of second kind
h	Thickness of shell
$i = \sqrt{-1}$	
J_n	Bessel Function of first kind
L	Length of longitudinal half-wave of mode of vibration
ℓ	Spacing of ring stiffeners of shell
m	Mass of shell per unit area
m_v	Virtual mass of entrained medium
N	Coefficient, see Eq. (6)
n	Number of circumferential waves of mode of vibration
$P, \overline{P}$	External radial force, see Eqs. (3), (11)
$p, \overline{p}$	Pressure in medium due to force P and $\overline{P}$, respectively
R	Subscript, refers to resonance
S	Force exerted by diaphragms
s	Defines spacing of diaphragms, Fig. 9

*Additional symbols in the appendices are defined as they occur.

t	Time
Y_n	Bessel Functions of second kind
α_k	Coefficient, Eq. (7)
β	Coefficient, see Eq. (9)
Δ	Coefficient, see Eq. (10) and Table A
δ	Damping coefficient
$\delta(x)$	Dirac's Delta Function
λ	Decay constant, Eq. (35)
ξ	Variable of integration related to L, Eq. (2)
ρ	Mass density of medium
Ω	Frequency of vibration
ω_k	Frequency of cylinder in vacuo

Synopsis

The purpose of the paper is the determination of the sound field due to a source inside a cylindrical shell which is submerged in an acoustic medium. Using results from a previous publication (Ref. 1) it is possible to express the radiation by a Fourier Integral; asymptotic integration for points at large distances furnishes surprisingly simple results.

While the fundamental theory applies for infinitely long unstiffened shells, it is extended with approximations first to stiffened shells, and as a final step in a qualitative manner to finite shells.

Particular attention has been given to the question of resonant frequencies i.e. those at which unusually high pressures may occur. For this purpose it became necessary to extend the theory to include the effects of structural damping; interesting conclusions concerning the radiation were obtained by considering the interaction between structural and radiation damping.

Numerical results giving typical pressure distributions at large distances were obtained for a lightly stiffened steel shell in water. It is found that such a shell has resonant frequencies with pronounced peaks of the sound pressure.

A detailed study on the effect of rigid diaphragms on the resonant cases was made. It was found that the addition of diaphragms reduces the resonant effects; for the steel shell selected as example diaphragms reduce resonant effects to an insignificant level.

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I. Previous Results on Forced Vibrations of Submerged Unstiffened Shells

The response of an infinitely long, thin cylindrical shell submerged in an acoustic medium due to sinusoidally distributed periodic radial forces was obtained in Ref. 1, using the modes of free vibrations of the shell in vacuo as generalized coordinates. For each pair of parameters n and ξ three frequencies ω_k exist, k = 1, 2, 3, the corresponding modes having longitudinal, tangential and radial displacements, respectively, Fig. 1,

$$\begin{aligned} u &= \frac{U_k}{W_k} \cos n\theta \cos \xi z \\ v &= \frac{V_k}{W_k} \sin n\theta \sin \xi z \\ w &= \cos n\theta \sin \xi z \end{aligned} \tag{1}$$

where U_k, V_k and W_k are numerical values which for certain ranges of n and ξ are tabulated in Ref. 2. The parameter ξ defines the half wave length L used in Refs. 1 and 2.

$$L = \left| \frac{\pi}{\xi} \right| \tag{2}$$

Let a space and time periodic external radial force, act on the submerged shell,

$$P(t, \theta, z) = \cos n\theta \, e^{i\xi z} e^{i\Omega t} \tag{3}$$

where the complex representation of P used here differs from, but is equivalent to the one in Ref. 1. The pressure p in the fluid at a point having the coordinates r, θ and z is

$$p = \frac{F(r)}{F(a)} \frac{\cos n\theta \, e^{i\xi z} e^{i\Omega t}}{1 - \dfrac{2m}{\rho a N \Delta}} \tag{4}$$

and the radial displacement w of the shell at a point having the coordinates θ and z is

$$w = \frac{\cos n\theta \, e^{i\xi z} e^{i\Omega t}}{m\Omega^2 \left[\dfrac{\rho a}{2m} \Delta - \dfrac{1}{N} \right]} \tag{5}$$

a is the radius of the cylinder, and ρ and m are the mass density of the acoustic medium, and of the shell per unit of area, respectively. N is a function of the elastic properties of the shell

$$N = \frac{\alpha_1 \Omega^2}{\omega_1^2 - \Omega^2} + \frac{\alpha_2 \Omega^2}{\omega_2^2 - \Omega^2} + \frac{\alpha_3 \Omega^2}{\omega_3^2 - \Omega^2} \tag{6}$$

N depends only on the three frequencies ω_k of the shell in vacuo having half wave length $L = |\pi/\xi|$. The coefficients α_k depend on the shape of the modes

$$\alpha_k = \frac{W_k^2}{U_k^2 + V_k^2 + W_k^2} \quad (k = 1, 2, 3) \tag{7}$$

and are also tabulated in Ref. 2. If n = 0 one of the coefficients α_k vanishes and Eq. (6) has only two terms.

The function F(r) in Eq. (4) arises in the solution of the wave equation for the acoustic medium, and is defined by Eq. (42) of Ref. 1 in different ways depending on the value of Ω. This is inconvenient for the present purpose and can be avoided by a modification. Noting that, for a real x, the Hankel Functions of the first and second kind are related $H_n^{(1)}(ix) = (-1)^{n+1} H_n^{(2)}(-ix)$ one obtains for the entire range

$$\frac{F(r)}{F(a)} = \frac{H_n^{(2)}\left(r\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right)}{H_n^{(2)}\left(a\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right)} = \frac{H_n^{(2)}(\beta r)}{H_n^{(2)}(\beta a)} \tag{8}$$

where the sign of the square root must be selected such that*

$$\beta = \sqrt{\frac{\Omega^2}{c^2} - \xi^2} \qquad [\mathrm{Re}(\beta) \geq 0,\ \mathrm{Im}(\beta) \leq 0] \tag{9}$$

Finally, the term Δ occuring in Eq. (4) and (5) is also a function of β,

$$\Delta = \Delta\left(a\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right) = -\frac{2}{a}\,\frac{H_n^{(2)}(\beta a)}{\frac{\partial}{\partial a} H_n^{(2)}(\beta a)} \tag{10}$$

Numerical values of Δ for real values of the argument will be required later. Such values are tabulated in Table A.

II. Unstiffened Shells Acted on by Forces Close to the Origin z = 0

The solution (4) will be utilized in this paper to find the pressure field due to radial periodic forces $\overline{P}(t, \theta, z)$ which are sinusoidally distributed in the circumferential direction, but non-periodic in the z-direction

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*Note that β as defined here corresponds to $\overline{\beta}$ in Ref. 1.

Table A. Values of Δ

$a\beta = a\sqrt{\frac{\Omega^2}{c^2} - \xi^2}$	n = 0	n = 1	n = 2	n = 3	n = 4
0	$\infty - \pi i$	2.000	1.000	0.667	0.500
0.1	4.774 - 3.052 i	2.048 - 0.032 i	1.003 - 0.000 i	0.667 - 0.000 i	0.500 - 0.000 i
0.2	3.339 - 2.879 i	2.135 - 0.131 i	1.010 - 0.0003i	0.669 - 0.000 i	0.501 - 0.000 i
0.4	1.974 - 2.479 i	2.250 - 0.530 i	1.045 - 0.005 i	0.676 - 0.0000 i	0.503 - 0.000 i
0.6	1.297 - 2.117 i	2.092 - 1.040 i	1.107 - 0.026 i	0.688 - 0.0002 i	0.508 - 0.000 i
0.8	0.908 - 1.820 i	1.684 - 1.383 i	1.196 - 0.082 i	0.708 - 0.0014 i	0.514 - 0.000 i
1.0	0.666 - 1.584 i	1.244 - 1.478 i	1.293 - 0.199 i	0.738 - 0.0051 i	0.523 - 0.0008 i
1.5	0.354 - 1.176 i	0.570 - 1.253 i	1.271 - 0.753 i	0.858 - 0.0547 i	0.559 - 0.0018 i
2.0	0.216 - 0.925 i	0.303 - 0.988 i	0.760 - 1.025 i	1.010 - 0.270 i	0.628 - 0.016 i
2.5	0.145 - 0.759 i	0.185 - 0.801 i	0.392 - 0.913 i	0.918 - 0.655 i	0.740 - 0.086 i
3.0	0.103 - 0.642 i	0.124 - 0.669 i	0.221 - 0.759 i	0.570 - 0.812 i	0.837 - 0.292 i
4.0	0.060 - 0.489 i	0.067 - 0.502 i	0.095 - 0.546 i	0.184 - 0.632 i	0.465 - 0.684 i
5.0	0.039 - 0.394 i	0.042 - 0.401 i	0.053 - 0.425 i	0.081 - 0.471 i	0.161 - 0.550 i
10.0	0.010 - 0.199 i	0.010 - 0.200 i	0.011 - 0.203 i	0.012 - 0.209 i	0.014 - 0.217 i

$$\overline{P}(t, \theta, z) = \cos n\,\theta\; g(z) e^{i\Omega t} \tag{11}$$

It is intended to consider forces acting close to the origin z = 0 such that

$$\int_{-\infty}^{\infty} |g(z)|\, dz$$

is finite. The restriction on the integral over g(z) excludes cases of forces periodically spaced in the z-direction.

Using the Fourier inverse of g(z)

$$G(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(z) e^{-i\xi z}\, dz \tag{12}$$

the loading function $\overline{P}(t, \theta, z)$ can be expressed by the Integral

$$\overline{P}(t, \theta, z) = \cos n\,\theta\; e^{i\Omega t} \int_{-\infty}^{\infty} G(\xi) e^{i\xi z}\, d\xi \tag{13}$$

This equation expresses $\overline{P}$ as an integral over a spectrum of forces of the type of Eq. (3), and the fluid pressure $\overline{p}$ due to the forces $\overline{P}$ can therefore be expressed by an integral over the spectrum of pressures, Eq. (4); substituting Eq. (8) the pressure $\overline{p}$ at a point having the coordinates r, θ, z is given by the integral

$$\overline{p} = \cos n\,\theta\; e^{i\Omega t} \int_{-\infty}^{\infty} \frac{H_n^{(2)}(\beta r)\, G(\xi) e^{i\xi z}\, d\xi}{H_n^{(2)}(\beta a)\left[1 - \frac{2m}{\rho a N \Delta(a\beta)}\right]} \tag{14}$$

It is noted that the integrand has branch points at $\xi = \pm\,\Omega/c$. On examination it is found that the restriction on β, Eq. (9), requires the path to pass above the branch point at $+\,\Omega/c$ and below at $-\,\Omega/c$. It is noted further that poles may occur on the real axis, i.e. on the path of integration. The required value of the integral is the principal one, i.e. it is the limit for $\epsilon \to 0$ of the sum of the integrals taken over the sections of the real axis obtained by excluding those portions of the real axis which are less than a small distance ϵ from the poles. The poles occur wherever the denominator bracket in Eq. (14) vanishes; by comparison with Ref. 1, Eq. (28), it is seen that the bracket is identical with the frequency equation for free vibrations, and poles occur at points where the forcing frequency Ω coincides with a frequency of free vibration of the shell having a half wave length $L = |\pi/\xi|$. It has been shown in Ref. 1 that free vibrations (Ω and ξ real) occur only if $\Omega \leq \pi c/L = |a\xi|$ and poles are therefore only possible for $|\xi| \geq \Omega/a$. The denominator

bracket being an even function poles must occur in pairs, at $\pm \xi_j$; there may be none, or up to three pairs depending on the value of Ω. A typical path of integration is shown in Fig. 2.

The integrand of Eq. (14) is a highly complicated function of the variable of integration ξ, and an exact evaluation of the formal solution (14) is out of the question. However, it is possible to evaluate the integral for large values of r in closed form as shown in Appendix A.

The asymptotic value of the pressure is

$$\bar{p} = \exp\left[i\left(\frac{\pi(n+1)}{2} - \frac{\Omega}{c}\sqrt{r^2+z^2}\right)\right] \frac{2\cos n\theta\, e^{i\Omega t}\, G(\xi_0)}{\sqrt{r^2+z^2}\, H_n^{(2)}(a\beta_0)\left[1 - \frac{2m}{\rho a N(L_0)\Delta(a\beta_0)}\right]} \tag{15}$$

where

$$\xi_0 = -\frac{z\Omega}{c\sqrt{r^2+z^2}} \tag{15a}$$

$$\beta_0 = \sqrt{\frac{\Omega^2}{c^2} - \xi_0^2} = \frac{r\Omega}{c\sqrt{r^2+z^2}} \tag{15b}$$

$$L_0 = \frac{\pi c\sqrt{r^2+z^2}}{\Omega z} \tag{15c}$$

The term $N(L_0)$ occuring in Eq. (15) is the value of N according to Eq. (6) for the half wave length L_0.

Due to its derivation Eq. (15) applies only for sufficiently large values of the radius r. To get at least a crude approximation r must be large enough to satisfy the two conditions (see Appendix A)

$$\frac{r}{a} > 50$$

$$\frac{r^2}{a\sqrt{r^2+z^2}} > \frac{(2n+1)c}{a\Omega} \qquad (\text{if } n \leq 6) \tag{15d}$$

Except for very low frequencies, or large ratios z/r, the first condition controls.

An alternative asymptotic evaluation for large values of z is feasible; it would essentially give information on the conduction of sound in the direction of the axis of the shell. This paper being aimed at other questions, the matter is not carried through.

If the applied load acts in the vicinity of $z = 0$ it is to be expected for physical reasons that the pressure at large distances will not depend greatly on the distribution of the

applied load $\overline{P}(t, \theta, z)$ in the z-direction and we consider therefore only the effect of a line load acting on the circle, $z = 0$,

$$\overline{P}(t, \theta, z) = \delta(z)\cos n\theta \, e^{i\Omega t} \tag{16}$$

where $\delta(z)$ is Dirac's δ-function: Eq. (12) gives

$$G(\xi) = \frac{1}{2\pi} \tag{17}$$

The peak values of the pressure, $|\overline{p}|$, for the case of a line load according to Eq. (16) becomes

$$|\overline{p}| = \frac{\cos n\theta}{\pi\sqrt{r^2 + z^2}} \left| \frac{1}{H_n^{(2)}(a\beta_0)\left[1 - \frac{2m}{\rho a N(L_0)\Delta(a\beta_0)}\right]} \right| \tag{18}$$

For z = 0 Eq. (18) simplifies further, as $\xi_0 = 0$ and $L_0 = \infty$ become independent of the frequency Ω,

$$|\overline{p}| = \frac{\cos n\theta}{\pi r} \frac{1}{H_n^{(2)}\left(\frac{a\Omega}{c}\right)\left[1 - \frac{2m}{\rho a N(\infty)\Delta\left(\frac{a\Omega}{c}\right)}\right]} \tag{19}$$

III. Application to Shells with Closely Spaced Ring Stiffeners

It was pointed out in Ref. 1, p. 15, that the results for the unstiffened shell, quoted in Sec. I, are applicable, approximately, to shells with closely spaced ring stiffeners, Fig. 3, if the frequencies of the stiffened shell in vacuo are used as values ω_k occuring in the expression for N, and if the mass of the stiffeners is averaged over the surface of the shell and included in the mass term m. The approximation can be expected to be good provided the length L of the half wave is several times the stiffener spacing ℓ, and provided the forcing frequency Ω is sufficiently low, such that the local displacement of the shell between the stiffeners is unimportant.

The results of the present paper were obtained by integration of contributions coming from all wave lengths $L = |\pi/\xi|$, including the range $L < \ell$, and this appears at first to preclude the application to stiffened shells. However, in the range where the integrand is a poor approximation,

$$\beta = \sqrt{\frac{\Omega^2}{c^2} - \xi^2}$$

is imaginary if Ω is sufficiently small, i.e. if $\Omega < \pi c/\ell$. If β is imaginary the term

$$\lim_{r \to \infty} \frac{H_n^{(2)}(\beta r)}{H_n^{(2)}(\beta a)} = \sqrt{\frac{a}{r}}\, e^{-(r-a)|\beta|} \tag{20}$$

in the integrand of Eq. (14) decays exponentially with increase in r, and the range in question gives no contribution to the pressure $\bar{p}$ at large distances. (Physically speaking this is so because these components of the shell motion do not radiate energy, see the discussion of forced vibrations if $\Omega < \pi c/L$, Ref. 1, p. 12.)

It is therefore, concluded that Eq. (15) is applicable as long as $\Omega < \pi c/\ell$, provided ℓ is small enough such that the local displacements of the shell between the stiffeners are unimportant.

Further consideration indicates that Eq. (15) may even be applicable for larger values of Ω, but only for locations where z/r is small. The essence of the process of asymptotic integration which leads to Eq. (15) is the fact that the pressure at any point is essentially affected by the radiation of a component of the shell motion of half wave length L_0, Eq. (15c). While no proof is available one can assume that the present results are meaningful for a stiffened shell provided $L_0 > \ell$, i.e. if

$$\Omega < \frac{\pi c}{\ell} \sqrt{1 + \frac{r^2}{z^2}} \tag{21a}$$

The additional requirement that the local displacements of the shell between the stiffeners are unimportant will be satisfied for a value n only if the respective lowest frequency ω_1 ($L = \infty$) of the shell in vacuo for $L = \infty$ is smaller than the "local" frequency of the shell if the stiffeners were rigid. Approximating the latter by the frequency of a clamped plate of span ℓ, one obtains

$$\omega_1(L = \infty) << 6.2\, \frac{hc_s}{\ell^2} \tag{21b}$$

where c_s is the velocity of sound in the shell material. It is further necessary that the forcing frequency Ω is smaller than the local frequency of the shell in the submerged state,

$$\Omega << 6.2\, \frac{hc_s}{\ell^2} \left(1 + \frac{\rho \ell}{\pi m}\right)^{-1/2} \tag{21c}$$

The last term is based on the approximate expression for the virtual mass $m_v = \rho\ell/\pi$.

In order to apply Eq. (15) in the suggested manner to ring stiffened shells of the type shown in Fig. 3, it is necessary to find the frequencies ω_k of such shells in vacuo. Suitable approximate methods for this purpose will be presented elsewhere.

The approximate treatment of stiffened shells in the manner indicated is tantamount to considering the stiffened shell as an orthotropic unstiffened shell. As n becomes larger the value $\omega_1(L = \infty)$ increases rapidly and the limitation Eq. (21b) indicates that the approach can only be used for low values of n. For high values of n the rings are relatively so rigid that they may be considered as absolutely rigid; this fact is utilized for an approximate treatment of resonant cases for large n, see Secs. VIe and VIIc.

IV. Discussion of the Results for Infinitely Long Shells

a. General

The most remarkable fact is that the term $N(L_0)$ in Eq. (15) expresses the entire effect of the elastic behavior of the shell on the pressure $\bar{p}$ at a point A, and more generally on the energy radiated in the direction OA, Fig. 4. The term N depends in turn only on the three frequencies ω_k and the corresponding modes in vacuo of the one half wave length $L_0 = |\pi/\xi_0|$, the three coefficients α_k being functions of the shapes of the respective mode. This half wave length L_0 is a function of the ratio r/z and of the frequency Ω, but not of the circumferential wave number n. For a given frequency Ω the radiation in any direction OA is only affected by the vibrations of the shell of half wave length L_0.

It is also noted that the pressure decays as the inverse of the distance from the origin, $\sqrt{r^2 + z^2}$. This decay is similar to the one found in the case of a pulsating force acting on a spherical shell. The decay differs fundamentally from the one found in Ref. 1 for a cylindrical shell acted on by forces which are space periodic (as $\sin \xi z$) in the direction of the longitudinal axis z; for such loads the decay was exponential and very fast in the non-radiating low frequency range, but as $1/\sqrt{r}$ for high frequencies. The difference can be attributed to the fact that in the latter case energy is radiated essentially only in two dimensions, while the presently treated case is three dimensional.

Eq. (18) was used to plot typical pressure curves for n = 0, 1 and 2 for line loads, Eq. (16), acting on a lightly stiffened* thin steel shell in water. Fig. 5 shows the pressure $\bar{p}$ versus the angle arc tg r/z. Due to the fact that r must be larger than a certain limit, Eq. (15d), the curve never applies for directions parallel to the axis of the shell. The pressure $\bar{p}$ and the radiation vanish for certain directions. The points $\bar{p} = 0$ can be traced to the fact that the term $N(L_0)$ in Eq. (18) vanishes. The case N = 0 is discussed in Ref. 1; it corresponds to a vibration damper situation in which the shell vibrates

*The rigidity of the stiffeners selected was such that the fundamental frequency for n = 2 for the mode of half wave length $L = \infty$, i.e. when the shell vibrates like a ring in vacuo, was 0.16 c/a. The forcing frequency was Ω = 0.8 c/a.

longitudinally and tangentially, while the radial displacements w vanish. The number of points for which $N = 0$ is one, or less for $n = 0$, and two or less, for $n \geq 1$, depending on the frequency. For loads which are not line loads, where Eq. (17) does not apply, additional points $\bar{p} = 0$ not included in the above statement occur wherever $G(\xi_0) = 0$.

b. Resonance

It is of considerable interest to examine the results for possible resonance at special frequencies. The following discussion applies to steel shells in water and may require modification for other media. The crucial term to consider is the expression in the bracket in the denominator of Eqs. (15) and (18). One can see immediately that this expression can never vanish, because the term $\Delta(a\beta_0)$ is complex for all admissible values of $a\beta_0$, while all other terms are real. However, in the range where the imaginary part of Δ is very small there is a possibility for very large magnification, if the real part of the value of the bracket should vanish in the same range. Table A indicates that for $n \neq 0$, $\mathrm{Im}\Delta$ is quite small for low values of the argument,* $a\beta_0 \leq n/2$. Eq. (15b) indicates that such low values correspond either to low values of Ω, $a\Omega/c \leq n/2$, or to large values of z/r; in either case large magnification indicating resonance will occur if the physical properties of the shell and medium are such that the real part of the value of the bracket vanishes or becomes very small. The approximation $\mathrm{Re}\,\Delta \simeq 2/n$, (good for $a\beta_0 \leq n/2$) leads to the condition

$$N(L_0) \cong \frac{nm}{\rho a} \qquad (n \neq 0) \tag{22}$$

If for a given low frequency, $\Omega \leq cn/2a$, a range of lengths $L_0 > \pi c/\Omega$ can be found which satisfies Eq. (22) there will be very large radiation; usually Eq. (22) will be satisfied, if at all, for any length $L_0 > \pi c/\Omega$, indicating large radiation in all directions.

It is noted that Eq. (22) is actually the frequency equation for the submerged shell (Ref. 1, Eq. (28)) if $\Delta \simeq 2/n$ is substituted. The real roots of this frequency equation all correspond to non-radiating lengths $L_0 < \pi c/\Omega$ and cause no pressure for large radii r. If the frequency equation has complex roots Ω with small imaginary parts, Eq. (22), being an approximation of the frequency equation, will be satisfied. Any large magnification found can, therefore, be ascribed to the proximity of a slightly damped natural frequency of the shell.

c. Steel Shells in Water

For the case of thin steel shells in water the criterion Eq. (22) indicates resonant cases for $n \geq 2$, but not for $n = 1$ (unless the mass of the shell is about twice or more

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*The limits for $\mathrm{Im}\Delta$ being small vary with n; for $1 \leq n < 6$ the value $n/2$ is reasonable, for larger n the range extends and for $6 \leq n \leq 15$ the limit is $\sim 2n/3$.

than the mass of the displaced water). For $n = 0$ no resonance with high magnification is possible as was pointed out before. Figs. 6a, b show the pressure $\bar{p}$ for $z = 0$ as function of the forcing frequency Ω for the shell used for Fig. 5. As expected, very narrow but extremely high peaks occur in the response for $n \geqslant 2$. Due to the fact that for low frequencies the arguments $a\Omega/c$ of the Hankel Functions and of the function Δ are quite small it was found necessary and convenient, to use in this range approximate values of these functions for small arguments rather than interpolate the tabulated values. Another aspect of the resonant cases is considered in the next section.

Fig. 6a demonstrates an important trend concerning the magnitude of the pressure $\bar{p}$ as function of the circumferential wave number n. It is apparent, that for very low frequencies the pressure for $n = 1$ is dominant, excepting resonant peaks for $n \geqslant 2$; it can be shown easily that for small arguments $a\Omega/c$ the pressure $\bar{p}$ is a linear function if $n = 1$, but a quadratic one for $n = 0$, and of the n^{th} power for the non-resonant ranges $n \geqslant 2$.

While Figs. 6a, b only contain cases $n < 5$, it can be shown for all values of n that the response $\bar{p}$ for low values of n, up to $n \cong 2a\,\Omega/c$ is several magnitudes larger than for $n \gtrsim 2a\,\Omega/c$, excluding points of resonance. If a force is considered which is not sinusoidally distributed over the circumference, it can be expanded in a Fourier series; the pressure is then also obtained in a Fourier series, in which only a limited number of terms will be significant. Fig. 7 shows the pressure in the plane $z = 0$ as function of the angle θ for a concentrated radial unit load acting at $\theta = 0$; only three terms of the Fourier series, $n \leqslant 2$, had to be used for $\Omega = 0.8$ a/c. Fig. 7 shows a very unexpected uniformity of radiation in all directions. Except for frequencies where resonance occurs for one of the components, this behavior seems representative for $a\Omega/c \gtrsim 0.5$. For smaller frequencies the $n = 1$ component is dominant and the radiation must vary as $\cos\theta$, giving a different picture.

d. Limitations Due to Stiffener Spacing

The results shown in Figs. 5-7 apply strictly only to an orthothropic shell, and only approximately to a ring stiffened one, see Sec. III. To find the limits of applicability to stiffened shells assume the stiffener spacing $\ell = 0.25a$, shell thickness $h = a/100$, the ratio $c_s/c = 3.5$, and examine the limitations Eqs. (21a, b, c).

Of the two limits for the frequency (21a, c) the first one gives $a\Omega/c < 12.5$, the second one, which controls in this case, $a\Omega/c << 2.7$; this indicates that the portion of Fig 6b for larger frequencies does not apply for the selected shell.

The limitation Eq. (21b) is of a different nature. It requires the frequency of the stiffening rings in vacuo for the particular value n to be below the limit $\omega_1(L = \infty) << 3.5$ c/a. As these frequencies increase with the number n, this limitation excludes high values of n. The ring stiffeners were selected such that for $n = 2$ the value $\omega_1(L = \infty) = 0.16$ c/a; the bending frequencies of a ring increase as $n(n^2 - 1)/\sqrt{n^2 + 1}$; the frequency for $n = 7$ being $\omega_1(L = \infty) \doteq 2.8$ c/a, the analysis does not apply for $n > 6$.

It follows that the results shown in Figs. 5-7 are valid also for stiffened shells,

except for the high frequency part of Fig. 6b. It is of interest to state that it seems possible to extend the theory, and eliminate at least the limitations Eqs. (21b and c). It will be seen in Sec. VIIb that even the present theory is applicable in the resonant cases in the range where Eqs. (21b, c) are violated.

V. Peak Pressures at Resonance, and Effect of Structural Damping

The largest resonant value of $\bar{p}$ for a given frequency Ω will occur approximately in the location z/r for which the value ξ_0, Eq. (15a), is such that the real part of the denominator bracket in Eq. 18 vanishes. Let this value be $\xi_0 = \xi_R < \Omega/c$. Consider the denominator D of Eq. (18),

$$D = H_n^{(2)} \left[1 - \frac{2m}{\rho a N \Delta}\right] \tag{23}$$

where $H_n^{(2)}$ and Δ are functions of

$$a\beta_R = a\sqrt{\frac{\Omega^2}{c^2} - \xi_R^2}\ .$$

Using Eq. (10) and the relation $H_n^{(2)} = J_n - iY_n$, where J_n and Y_n are Bessel Functions of the first and second kind,

$$D = J_n - \frac{a\beta_R m}{\rho N} J_n' - i\left(Y_n - \frac{a\beta_R m}{\rho N} Y_n'\right) \tag{24}$$

In the resonant range the functions Y are very much larger than the J, and the vanishing of the last term causes resonance. Using this condition,

$$Y_n - \frac{a\beta_R m}{\rho N} Y_n' = 0,$$

to eliminate N in the first term of Eq. (24) gives

$$D = \frac{J_n Y_n' - J_n' Y_n}{Y_n'} = \frac{2}{\pi a\beta_R Y_n'} \tag{25}$$

where the relation on the Wronskian (Ref. 3, p. 76) was used. The pressure $\bar{p}$ is finally

$$\frac{\pi\sqrt{r^2+z^2}}{\cos n\theta}\left|\bar{p}_R\right| = \frac{\pi a\beta_R}{2}\left|Y_n'(a\beta_R)\right| \tag{26}$$

As the values Y' are very large if the argument is much smaller than the order n, this pressure is much higher than the one at the same point at a slightly different

frequency; the order of magnitude of the latter can be obtained by noting that the bracket in Eq. (23) in this range is of the order of unity. With $H_n^{(2)} \sim Y_n$

$$\frac{\pi\sqrt{r^2+z^2}}{\cos n\theta}\left|\bar{p}\right| \simeq \frac{1}{Y_n(a\beta)} \tag{27}$$

The ratio between the values of Eq. (26) and (27), i.e. the magnification, can be enormous if $a\beta_R \ll n$. This fact makes it necessary to consider the effect of structural damping. It can be introduced approximately* in the previous computations by replacing the term $1/N$ wherever it occurs by $(1 + i\delta)/N$. The effect of damping in general is found to be insignificant, except in the resonant cases. Proceeding from Eq. (24) to find the new value of $\bar{p}_D$, one obtains

$$\frac{\pi\sqrt{r^2+z^2}}{\cos n\theta}\left|\bar{p}_D\right| = \left|\frac{Y_n'}{\frac{2}{\pi a\beta_R} - \delta Y_n Y_n'}\right| \tag{28}$$

where small terms J_n were neglected versus Y_n terms. For not too large arguments $Y_n Y_n'$ will be so large that the term $2/\pi$ is insignificant and

$$\bar{p}_D = \frac{1}{\delta Y_n} \tag{29}$$

There is still a magnification, $1/\delta$, but as $1/Y_n$ is very small the actual pressure level becomes unimportant compared to those for other values n. However, if the resonance occurs for a value $a\beta_R$ close to n, but still such that $\mathrm{Im}\Delta$ is small, the situation is different and high pressures are possible despite damping.

An estimate for the maximum value of $\bar{p}_D$ is obtainable. For $n \gg a\beta$

$$Y_n'(a\beta) = -\frac{n}{a\beta} Y_n + Y_{n-1} \simeq -\frac{n}{a\beta} Y_n$$

is a reasonably good approximation; substituting, Eq. (28) has a maximum for

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* The correct way to introduce structural damping is to introduce damping terms $2\delta_k \omega_k q_k$ in the equations of motion (Ref. 1, Eq. (34)), where $\delta_k > 0$ is the ratio of critical damping. This leads to a new expression replacing the term N,

$$N_D = \sum_{k=1}^{3} \frac{\alpha_k \Omega^2}{\omega_k^2 + 2i\delta_k \omega_k \Omega - \Omega^2}$$

As Ω, α_k and ω_k^2 are necessarily positive it is easy to show that $1/N_D$ must have a positive real part and one can use the approximation $1/N_D \sim (1 + i\delta)/N$.

$$Y_n = \sqrt{\frac{2}{n\pi\delta}}$$

Due to the approximation this is an upper limit for the actual value of $\bar{p}_D$.

$$\frac{\pi\sqrt{r^2 + z^2}}{\cos n\theta} \max\left|\bar{p}_D\right| < \frac{1}{2}\sqrt{\frac{n\pi}{2\delta}} \tag{30}$$

The maximum values of $|\bar{p}_D|$ increase for a given δ with $\sqrt{n}$. In order to judge their importance they should be compared not only to the values of Eq. (27) for the same number n, but also to the values of the pressure $\bar{p}$ for the frequency Ω but different values of n. This changes the picture because for a given frequency $a\Omega/c$ the modes $n \lesssim a\Omega/c$ have much larger values $\bar{p}$ than those where $n > a\Omega/c$, particularly if $a\Omega/c$ is large.

The level of the pressures for high frequencies in the lower modes can be estimated by the following approximation: if the frequency Ω is sufficiently large, $a\Omega/c >> n$, the effect of the elastic resistance of the structure vanishes and essentially only the effect of the mass of the shell remains; this expresses itself in the previous equations by $N \to -1$. Using the asymptotic expression Eq. (A-3) gives the comparitive level of nonresonant pressures

$$\lim_{\Omega \to \infty} |\bar{p}| \sim \frac{\rho a}{2m}\sqrt{\frac{2\pi}{a\beta}} \tag{31}$$

The peak value, Eq. (30), occurs in the vicinity of $a\beta = n$, and it is seen immediately that the pressure given by Eq. (30), at least for sufficiently large n, will exceed the value of Eq. (31).

Before discussing the results further, attention is drawn to the fact that Eqs. (26), (28) and (30) give the pressure $\bar{p}$ for any frequency Ω for which a resonance occurs. The value of the frequency, if any, for which this happens does not follow from these equations, but must be found from Eqs. (18, 19). This means, even if a resonance occurs for a given shell it may be at a point where Eq. (28) gives a very much lower value than the limit, Eq. (30).

To illustrate the effect of structural damping on the pressures $\bar{p}$ at resonance, Figs. 8a, b show the values of $\bar{p}$ at resonance without damping, and with structural damping $\delta = 0.01$, for $n = 2$ and 10. For comparison the level of nonresonant pressures for the steel shell considered in Sec. IV is also shown. In spite of the low structural damping value selected the change for low values $a\beta_R$ is revolutionary. While the undamped value goes to ∞ if $a\beta_R \to 0$, the damped values vanish if $a\beta_R \to 0$. As $a\beta_R$ approaches n the two curves merge and damping becomes unimportant. The damped curve has a peak, approximately given by Eq. (30).

The enormous effect of low structural damping for $a\beta_R << n$ can be interpreted physically. As $a\beta_R \to 0$, radiation damping goes to zero, while the structural damping remains finite. As the energy input will divide in the ratio of the damping terms, all

energy is absorbed by the structure and the radiation pressure $\bar{p}$ must vanish.*

The large effect of structural damping for small values of $a\beta_R$ eliminates the effect of resonances unless $\xi_R \sim 0$. Let resonance occur for a value ξ_R close to Ω/c; in such a case the value of $a\beta_R = a\sqrt{\Omega^2/c^2 - \xi_R^2}$ will be very small and the radiation insignificant, unless the resonant frequency had been extremely large, $\Omega >> n\ c/a$. The latter is, however, impossible if the velocity of sound in the shell and in the medium are of the same order. The conclusion can, therefore, be drawn that resonances can only be significant if $\xi_R \sim 0$.

Conclusion

The essential meaning of Figs. 8a and b is the fact that the interaction of structural and radiation damping allows substantial radiation only in a relatively narrow band of frequencies; its location and width depends on the value of n. The highest radiation occurs if structural and radiation damping are about equal. Only if the resonant frequency of the structure lies in this band can noticeable resonant effects ever occur; whether these resonant effects actually are noticeable depends in any particular case on the pressure level in the other modes, and can not be decided in a general manner. For the steel shell considered earlier this question is considered in Sec. VII.

VI. Displacements at Resonance and Effect of Rigid Diaphragms

a. Displacements

Applying the reasoning which led to Eq. (14) for the pressure $\bar{p}$, Eq. (5) leads to an expression for the radial displacement w due to the loading Eq. (16)

$$w = \frac{\cos n\,\theta\, e^{i\Omega t}}{2\pi m \Omega^2} \int_{-\infty}^{+\infty} \frac{e^{i\xi z} d\xi}{\frac{\rho a}{2m}\Delta(a\beta) - \frac{1}{N}}$$

The path of integration is the same as for the integral Eq. (14), see Fig. 2.

In general one may expect that an oscillating load of the type defined by Eq. (16), i.e. acting at $z = 0$, will result in displacements which are significant only for a "reasonable" distance from the force, comparable to the physical dimensions of the shell (radius a, shell thickness h). The integral in Eq. (32) was evaluated for resonant cases in Appendix B, in order to show that this is not generally true for the response of the shell at resonance. The response may then be extremely widespread.

The result of the analysis in Appendix B, Eq. (B-35), gives the displacement w

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*From this reasoning it would appear that the nearly complete suppression of radiation by structural damping in similar circumstances should occur not only in the case of cylindrical shells, but quite generally for any type of submerged structure.

for resonant cases $n \geq 2$ in the form *

$$w(z) = w(o)\, e^{i\xi_I |z|} \tag{33}$$

where $a\xi_I$ is in certain cases small versus unity. Consider a case $\xi_R = o$ where $a\Omega/c \ll n$; ξ_I is given by Eq. (B-33) and due to $Y_n \gg 1$ one obtains $|a\xi_I| = n(n-1)\delta$; for small damping δ and small n, this gives a very small value of ξ_I.

The fact that the amplitudes of the displacement vary only very slightly at distances $z = a$ or even more, raises the question of the effect of a rigid diaphragm located at some distance $z = sa$ from the load. In general, the displacement w, Fig. 9a, will be localized, such that a diaphragm does not interfere with the displacements noticeably; no appreciable effect on the radiation can be expected. This can no longer be said if the displacement is not localized, i.e. if, at resonance, the exponent ξ_I in Eq. (33) happens to be small. One will suspect that the diaphragms will not only prevent the displacement (33), but in doing so also reduce the unusually high radiation to normal levels.

The knowledge of the displacement, Eq. (33), makes it possible to treat the shell with one or more diaphragms by applying the principle of superposition, and obtain the effect of diaphragms on the radiation for the resonant cases.

b. Effect of Diaphragms

The effect of a "rigid" diaphragm, or stiffening ring, Fig. 9b, at a distance sa from the load $\overline{P}$, Eq. (16), is essentially the application of a load $S \cos n\theta\, e^{i\Omega t}$ at $z = sa$ such that the displacement at $z = sa$ vanishes.** The magnitude, i.e. the absolute value and the phase of S can be determined from this condition. Observing that the new force is of similar nature as $\overline{P}$, Eq. (33) gives also the deflections due to S, if z is replaced by $z - sa$ and one obtains

$$S = -e^{\lambda} \tag{34}$$

where

$$\lambda = ia\xi_I s \tag{35}$$

For two symetrically spaced diaphragms, Fig. 9c, the forces at each of them are

$$S = -\frac{e^{\lambda}}{1 + e^{2\lambda}} \tag{36}$$

$(i\xi_I)$ has a negative real part, which makes $|s| < 1$.

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*Note that w(o) is a function of t and θ, it contains a factor $\cos n\theta\, e^{i\Omega t}$.

**This is not true for $n = 1$ where the displacement is such that the shell sections remain circular and diaphragms are ineffective.

Consider now the asymptotic sound field due to the superposition of the original force $\overline{P}$ and the reactions S due to the diaphragms. For points at large distances r the differences sa in location of the forces $\overline{P}$ and S will be immaterial, and the entire effect is that of a force $\overline{P}_s = \overline{P} + \Sigma S$ at z = 0. For the case of one diaphragm,

$$|\overline{P}_s| = \overline{P}\,|(1 - e^{\lambda})| \simeq |\lambda|\overline{P}$$

and for two diaphragms

$$|\overline{P}_s| = \overline{P}\left|1 - \frac{2e^{\lambda}}{1 + e^{2\lambda}}\right| \simeq \left|\frac{\lambda^2}{2}\right|\overline{P} \tag{37}$$

where the approximations may be used only if $|\lambda|$ is sufficiently small. If λ is small the reduction in the force from $\overline{P}$ to $\overline{P}_s$, and the corresponding reduction of the sound pressure is substantial. The reduction for two diaphragms, being of the order λ^2, is much larger than for one only ; other arrangements of two or more diaphragms, Fig. 9d, can easily be studied; the reduction remains always of the order of λ^2.

c. Reduction of the Resonant Pressure by Diaphragms, no Structural Damping

For small frequencies $a\Omega/c \lesssim n/2$ Eqs. (26, 30) give the resonant pressure

$$\frac{\pi\sqrt{r^2 + z^2}}{\cos n\,\theta}\left|\overline{p}_R\right| \sim \frac{\pi n}{2}\,Y_n \tag{38}$$

For the case $\xi_R \sim 0$ Eq. (B-33) gives

$$|a^2\xi_1^2| = \frac{\frac{4(n-1)}{\pi}}{Y_n^2} \;;$$

the reduction of the force, Eq. (37), reduces the pressure to $|\overline{p}_s| = \frac{\lambda^2}{2}\left|\overline{p}_R\right|$, where λ is given by Eq. (35), and

$$\frac{\pi\sqrt{r^2 + z^2}}{\cos n\,\theta}\,|\overline{p}_s| = n(n-1)s^2\,\frac{1}{Y_n} \tag{39}$$

Y_n being very large, this is a considerable reduction; comparison with Eq. (27) shows that the magnification $n(n-1)s^2$ is only moderate, unless n or s are large; comparison with Eq. (29) shows that the reduction due to diaphragms for small n and $s \sim 1$ is larger than the one due to small structural damping. However, for larger n structural damping will be more effective than diaphragms.

For the case $\xi_R >> 0$, Eq. (B-13) gives $\xi_1 \sim \xi_R$; ξ_1 being no longer small, the analysis in Appendix B is invalid. It is, however, obvious without further analysis that, the radiation being emitted by the mode of half wave length $L = \pi/\xi_R$, only diaphragms spaced at $as \lesssim L/2$ can possibly be effective, if $a\xi_R \sim n$ this requires $s < \pi/2n$. Unless

such close diaphragms are used, damping will in this case be the dominant effect.

d. Reduction of the Resonant Pressure by Diaphragms, including Structural Damping

The point of principal interest is whether the peaks of the resonant pressure curves with damping, Fig. 8, can be further reduced by diaphragms of reasonable spacing, $s \sim 1$. Little further reduction below the level with damping can be expected if $\xi_R >> 0$, or if n is large, even if $\xi_R \sim 0$, as the preceding discussion clearly shows that damping in these cases is relatively much more effective than the diaphragms.

The evaluation shows that for n = 2 the resonant pressures with damping are substantially reduced by diaphragms; they are, however, reduced very slightly only, less than 10%, from the pressures with diaphragms but no damping. For n = 10 the situation is reversed. The pressures with damping and diaphragms are considerably less than those with diaphragms only, but the reduction is only slight, about 25%, if the results are compared with the damped case without diaphragms.

e. Required Rigidity of Diaphragms

The effect of "rigid" diaphragms or stiffening rings, was studied in this section and the question arises when is a physical member stiff enough to be considered rigid. Common sense will suffice to make the decision for diaphragms, but this is not true for stiffening rings. The decision depends on the relative values of the frequency of the ring, of the forcing frequency Ω, and of the frequency of the shell between the rings. The situation is similar to the one encountered in Sec. III where conditions were given to ensure sufficient stiffness of the shell. Using the symbols defined for Eqs. (21b, c), stiffening rings spaced at distances $\ell = 2sa$ may be considered "rigid" if

$$\Omega << \sqrt{1 + \frac{\rho \ell}{\pi m}}\, \omega_1(L = \infty) \tag{40a}$$

$$\omega_1(L = \infty) >> 6.2\, \frac{hc_s}{\ell^2} \tag{40b}$$

It is interesting to note that the frequency in vacuo $\omega_1(L = \infty)$ increases with increasing n, and that any stiffening ring for sufficiently large n may be considered rigid. Eqs. (21b) and (40b) differ only by the direction of the inequality, and it appears that for low n a shell with ring stiffeners can be treated by the approximate method of Sec. III. For large n the same rings can be considered "rigid"; the structure can then be studied first as a shell without stiffeners, with a subsequent correction for "rigid" stiffeners. While the latter case has not been solved generally, the resonant cases can be treated as shown in this section.

f. Conclusion

The effect of diaphragms or rigid stiffeners on the resonant pressures can only

be large for low values of $n \geqslant 2$, for high values of n any effect is insignificant compared to structural damping. The delineation of high and low values of n depends on the spacing of the diaphragms. For $s \sim 1$, i.e. a spacing about equal to the diameter, $n = 10$ is already high.

Even for low values of n the effect of diaphragms is not large, unless ξ_R is small. This is, however, of little importance as in the case of large ξ_R damping alone practically eliminates the effect of resonance, as was explained at the end of Sec. V.

The effectiveness of diaphragms in reducing radiation at resonance will be seen on the numerical example in the next section.

VII. Pressures at Resonance for a Typical Stiffened Steel Shell

The preceding two sections discussed the effect of damping and diaphragms in a rather abstract manner. In this section the actual resonant frequencies and the associated pressure levels $\bar{p}$ will be considered for a typical shell giving numerical results which may be easier to appreciate.

The properties of the shell are: $\rho a/2m = 4$, $h = a/100$, stiffener spacing $\ell = 0.25a$, diaphragms at distances of $3a$, making $s = 1.5$, Fig. 10, structural damping $\delta = 0.01$. The rigidity of the ring stiffeners is such that for $n = 2$ the frequency in vacuo is $\omega_1(L = \infty) = 0.16$ a/c.

In the last part of Sec. IV it was shown that the analysis of the stiffened shell is only applicable if $a\Omega/c << 2.7$ and $n < 7$. This range can be studied with the formulae developed so far. The range can be extended above these limits as will be shown later in this section.

a. Range $a\Omega/c << 2.7$ and $n < 7$

The resonant frequencies can be located from Eq. (22) which is the frequency equation for the submerged shell with $\Delta = 2/n$; for $\xi_0 = 0$ this is equivalent to the frequency equation for a ring carrying in addition to the mass m of the shell also the virtual mass of fluid (see Ref. 1, p. 12)

$$m_v = \frac{\rho a}{2} \alpha_1 \Delta \sim \frac{n \rho a}{n^2 + 1} \tag{41}$$

where $\Delta \sim 2/n$ and $\alpha_1 \sim n^2/n^2 + 1$. The frequencies of a ring in vacuo vary as

$$\frac{n(n^2 - 1)}{\sqrt{n^2 + 1}},$$

and as the inverse of the square root of the mass. Tying into the value 0.16 a/c in vacuo for $n = 2$ gives the resonant frequencies

$$\frac{a\Omega}{c} = \frac{0.060\, n(n^2 - 1)a}{c\sqrt{n^2 + 1 + \frac{n\rho a}{m}}} \tag{42}$$

Numerical values of Ω are listed in Table I. It is noted that Eq. (22) in this case has no root for $n = 1$; there are therefore no resonant frequencies for $n = 0$ or 1.

Using Eqs. (26, 28, 35, 37, and B-33) the resonant pressures with and without damping and diaphragms were computed (with slide rule accuracy) for the case $\xi_R = 0$, i.e. at $z = 0$, and are shown in Table I. All frequencies are sufficiently low so that the approximation Eq. (B-30) could be substituted into Eqs. (26, 28).

Table I. Pressure $|\bar{p}|$ at Resonance at $z = 0$

n	Resonant Frequency $a\Omega/c$	$-Y_n$	$\pi r\,|\bar{p}|/\cos n\theta$ = no damping no diaphragms	with damping no diaphragms	no damping with diaphragms	with damping and diaphragms
2	0.08	200	630	0.50	0.022	0.022
3	0.25	330	780	0.30	0.041	0.041
4	0.52	426	2680	0.24	0.063	0.063
5	0.9	435	3420	0.25	0.103	0.103
6	1.4	358	3370	0.28	0.19	0.19

The diaphragms are clearly dominant in determining the level of the pressure in this range as the figures with diaphragms and damping are indistinguishable from those with diaphragms but no damping.

To judge the importance of the pressure $\bar{p}$ reduced by the diaphragms it should be compared with the pressures $\bar{p}$ for the other values of n for the same frequency, Fig. 6a. It must be kept in mind that only the total pressure comprising the sum of the components for all values of n can be observed, and that a peak, say in the $n = 6$ component, will be lost in the total pressure, if this peak was small in comparison to the pressure in the modes $n = 0$, 1 and 2. This is the position in this range for all n in case of the shell with diaphragms; it can be predicted that regardless of the exact nature of the applied force the total pressure will not show appreciable signs of resonant frequencies. (This statement excludes the unlikely case of a force which contains only the n^{th} component.)

If the shell has no diaphragms the position is different; the case $n = 2$ would produce a quite pronounced resonance, as the level of the pressure is about 10 times larger than the largest of the other components. The result becomes then very sensitive to changes in the value of structural damping.

Table I also shows that with increasing n the resonant pressure levels increase

rapidly. This is in part due to the fact that the diaphragms become less effective for larger n. The damping effect is also peculiar, as the pressure level decreases from $n = 2$ to 5, but then increases again; this is due to the shape of the curves in Figs. 8. The peak of the damped pressure curve moves gradually towards increasing values of n, while the resonant frequency increases much faster, nearly like n^2. The resonant frequency therefore gradually moves into the peak of the response curve, which accounts for the increase of $|\bar{p}|$ for $n > 5$. Important resonances might occur for the shell under consideration above $n = 6$ requiring further consideration.

b. Refined Theory for Higher Values of Ω and n

The three limitations Eq. (21a - c) originate from the basic assumption that the deflection of the shell between stiffening rings must be negligible compared to the displacements of the rings. It is possible to approach the problem without this assumption in the following manner.

Let the deflection of the shell between stiffening rings be proportional to $1 - \cos 2\pi z/\ell$; the entire deflection is then

$$w = \cos n\theta \, e^{i\Omega t} \sin \xi z \left[W + B\left(1 - \cos \frac{2\pi z}{\ell}\right)\right] \tag{43}$$

where B is an arbitrary value. Provided the distance between nodes in the longitudinal direction, $\ell/2$, is smaller than the distance between circumferential nodes, $\pi a/n$, the shell in vacuo will vibrate between stiffening rings with a frequency roughly that of a clamped plate of span ℓ

$$\omega_s \sim 6.2 \frac{hc_s}{\ell^2} \tag{44}$$

The above condition on the nodes requires

$$n << \frac{2\pi a}{\ell} \tag{45}$$

For the above example this gives $n << 8\pi$, a considerable extension of the range.

Use is now made of the fact that each resonant frequency Ω is essentially the real part of a complex frequency of free vibration of the submerged shell; the resonance being pronounced only if the imaginary part of the frequency is small. The real parts of slightly complex frequencies of free vibrations can be found as frequencies of the shell in vacuo if the actual mass is increased by an appropriate virtual mass of the medium (Ref. 1, p. 10-12). The problem of finding the resonant frequencies is therefore reduced to the one of finding frequencies of a stiffened shell in vacuo having a slightly different mass (This approach was already used in deriving Eq. (42)).

Having made the assumption (43) concerning the deflections of the shell between

the stiffeners, the combined system of shell and rings can be treated by the method of Ref. 5. The resonant frequencies $\bar{\Omega}$, including the effect of the shell deflection, are the roots of the equation

$$\left(\frac{\Omega^2}{\bar{\Omega}^2} - 1\right)\left(\frac{\Omega_s^2}{\bar{\Omega}^2} - 1\right) = \frac{2}{3}\,\frac{m + m_v - \frac{m_R}{l}}{m + m_v} \tag{46}$$

where Ω is the resonant frequency if the shell is rigid, Eq. 42; Ω_s is the frequency of the shell vibrating between stiffeners, Eq. 44, but with a correction for the effect of the virtual mass

$$\Omega_s = \omega_s\sqrt{\frac{m - \frac{m_R}{l}}{m_v + m - \frac{m_R}{l}}} \tag{47}$$

m_R is the mass of the stiffeners per unit length; m_v is given approximately by Eq. (41).

To defend the procedure it is necessary to consider the interaction problem between shell and medium, treated in Ref. 1 for the case without local shell deformation. The entire displacement Eq. (43) may be written in the form

$$w = \cos n\,\theta\, e^{i\Omega t}\left[\bar{W}\sin\xi z + B_1\sin\left(\xi + \frac{2\pi}{l}\right)z + B_2\sin\left(\xi - \frac{2\pi}{l}\right)z\right] \tag{48}$$

where $\bar{W}$, B_1 and B_2 are new constants. This equation being a sum of sine terms in ξ the fundamental procedure of Ref. 1 can be applied, but there will be coupling between the modes having the parameters ξ and $\xi \pm 2\pi/l$. While this complicates matters it can be expected not to change the results qualitatively.

As far as resonant radiation is concerned it should be noted that if $\xi = \xi_R \sim 0$ and l is small the values $\xi \pm 2\pi/l$ will be so large that these modes do not radiate at all, and the intensity of radiation will be given by the various formulae derived earlier. This can be expected if the condition

$$\frac{a\Omega}{c} \leqslant \frac{a\pi}{l} \tag{49}$$

is satisfied in which case the suggested procedure will give particularly good results.

It is therefore believed that the results concerning resonant pressures obtained in Secs. V and VI remain qualitatively applicable if the resonant frequencies are obtained from Eq. (46) provided Eqs. (45) and preferably also (49) are satisfied.

The two resonant frequencies $\bar{\Omega}_{1,2}$ of the shell considered were obtained from Eq. (46) and are shown in Table II; for comparison the frequency Ω neglecting the effect of shell deflection between rings is also shown.

Table II. Resonant Frequencies of Stiffened Steel Shell

n	2	3	4	5	6	7	8	9	10	11	12	13
$a\Omega/c$	0.08	0.25	0.52	0.90	1.40	1.95	2.65	3.5	4.4	5.4	6.6	7.9
$a\bar{\Omega}_1/c$	0.08	0.25	0.51	0.86	1.23	1.61	1.86	2.10	2.28	2.38	2.50	2.65
$a\bar{\Omega}_2/c$	2.7	3.0	3.3	3.7	4.1	4.6	5.3	6.5	7.1	9.3	11.1	13.0

For low n the revised frequency equals the original one, $\bar{\Omega}_1 \sim \Omega$; the second frequency $\bar{\Omega}_2$ which did not appear in Table I is so high that it lies outside the range of pronounced resonance $a\Omega/c \lesssim n/2$. For $n > 6$, the lower values $\bar{\Omega}_1$ lie in the range where damping practically prevents radiation, as can be seen from Fig. 8b for n = 10. The higher frequencies $\bar{\Omega}_2$, on the other hand, are important because they lie for $n \sim 8$-10 in the peak zone of the pressure curve with damping, Fig. 8b. For larger values of n the value $a\Omega_2/c$ is larger than n and no pronounced resonance is possible.

In the range $n \sim 10$ diaphragms reduce the resonant pressure only slightly and relatively high resonant pressures in the components n = 8 to 10 must be expected at their respective resonant frequencies. For $\delta = 0.01$, which is quite a low damping value, the level would be about 5 times that of the other components for smaller n. However, if one takes into account that there are many of the other components which will add up, and that the expansion coefficient of the force for high n is usually lower than for low n, one can not expect pronounced signs of these resonances in the total pressure. There might be a doubling of the pressure, but no high peak. If the damping is higher than the assumed $\delta = 0.01$, the effect would even be less.

The refined analysis just discussed is subject to the limitations Eqs. (45) i.e. in the presently discussed case $n < 8\pi$; for the range of significant results $n \sim 8$-10, limitation (49) giving $a\Omega/c < 4\pi$ is also satisfied. This indicates that resonances for $n < 8\pi$ have essentially been covered.

c. The Range $n >> 8\pi$

For high values of n the approach mentioned in Sec. VIe can be used; the stiffening rings are for high values n so rigid that they may be treated as diaphragms. The resonant frequencies of the actual stiffened structure are then those of the shell without any stiffeners, only the magnitude of the resonant pressures being affected by the existence of the "rigid" stiffeners.

Considering the case $\xi_R \sim 0$, the resonant frequencies of the shell are those of a ring of thickness h; for large n the virtual mass of the medium becomes negligible and the frequency will be approximately equal to that of a flat plate having a span $\bar{l}$ equal to the distance between the nodal generators, $\bar{l} = a\pi/n$,

$$\frac{a\Omega}{c} \sim 0.27\, n^2 \frac{hc_s}{ac} \tag{50}$$

The resonant pressure $|\bar{p}_D|$ allowing for damping for large values of n is a function of the frequency similar to the one shown in Fig. 8b for n = 10, except that the peak will be closer to n and narrower. This means only those resonances will not be eliminated by damping for which the value $a\Omega/c$ given by Eq. (50) is slightly less than n. The interesting values of n are therefore

$$n \sim 3.5 \frac{ac}{hc_s} \tag{51}$$

For the case under discussion a/h = 100, c_s/c = 3.5, giving n = 100 and $a\Omega/c \sim 100$. Due to the high value of n, the distance between nodal lines, $a\pi/100$, is so much smaller than the distance between the ring stiffeners $\ell = a/4$, that the rings can have no appreciable effect. The expected resonant pressure can, therefore, be estimated by Eq. (30),

$$\frac{\pi r}{\cos n\theta} \left|\bar{p}_D\right| < \frac{1}{2} \sqrt{\frac{n\pi}{2\delta}} \sim 62 \tag{52}$$

The pressures $\bar{p}$ of the other components have a level given by Eq. (31), for the selected shell $\bar{p} \sim 1$. As there are nearly a hundred of these, the total pressure will again not show a pronounced sign of the resonance.

d. Conclusion

Except for transition points the entire range has been covered, but no indication of pronounced resonance in the pressures at large distances were found for the steel shell in water selected as example. Mild resonant effects increasing the total pressure (to maybe twice its regular value) can be expected in the range $a\Omega/c \sim 6$ and again $a\Omega/c \sim 100$.

The analysis did not indicate that the results are sensitive to changes in the dimensions of the shell; no pronounced resonant effects should therefore be expected for any shell of this type.

VIII. Resonances for n = 1

Eq. (22), which gives the location of resonant frequencies, has in this case no solution at all for the steel shell in water considered as example. One can find such solutions for steel shells with thicker walls, but only for much heavier shells, if $\rho a/2m \leq 1/2$. However, even these solutions pertain to values ξ_R which are nearly equal to Ω/c, i.e. to cases which are for practical purposes eliminated by damping, as explained at the end of Sec. V.

In the case n = 1 no solution of Eq. (22) exists ever for steel shells at or near $\xi_R = 0$, while for $n \geq 2$ solutions could be found near $\xi_R = 0$; these latter solutions

caused the widespread and relatively pronounced resonances for the shells without diaphragms. The different behavior for $n = 1$ is caused by the fact that for $n \geq 2$ the submerged shell has a slightly complex, low frequency of free vibration associated with a mode in which the generators of the cylindrical shell remain straight; the corresponding frequency for $n = 1$ degenerates into $\Omega = 0$, a rigid body motion of the shell, and the resonance disappears.

If one introduces in some way a restoring force which would resist rigid body motions of the shell, the trivial frequency $\Omega = 0$ would disappear and resonances in the mode $n = 1$ may occur. There is an interesting possibility of introducing such a restoring force. Consider a floating rigid shell with a mass center below the center line of the cylinder in a gravitational field; the shell can execute oscillations (rolling) during which the circular cross-section, Fig. 11, will be displaced laterally, coupling the body motion with the $n = 1$ oscillations of the fluid. Such a system will have resonances of the type found previously for $n \geq 2$. This is presumably also true if, instead of the rigid cylinder, an elastic shell with eccentric mass is considered. It may well be that the resonant radiation in this case would be quite high, as diaphragms will have no effect;* furthermore, the rigid body part of the motion of the shell will have no structural damping. This is replaced by the viscous damping due to the relative motion of shell and medium, but may be very low.

IX. Qualitative Validity of Results for Long Shells of Finite Length

If the displacements of an infinitely long shell due to an exciting force are localized as shown in Fig. 9a, one can reason that the sound field can not be affected materially if parts of the shell at large distances from the force are replaced by the acoustic medium. The results found for the infinite shell can, therefore, be applied to long but finite shells, if the load does not act close to the end, except in those cases where the displacements are not localized. This excludes certain resonant cases, and also the response in the $n = 1$ mode for low frequencies.**

If the shell has a number of rigid diaphragms one can reason further that their

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* See footnote on page 15.

** The displacement of an infinitely long bar in vacuo due to a transverse oscillating force is $w = e^{-\lambda z/a}$ where

$$\lambda = \sqrt[4]{\frac{EI}{a^4 m \Omega^2}} .$$

For sufficiently small frequency Ω the decay constant λ will become small, and the displacement can no longer be considered localized. (E is Young's modulus, I the moment of inertia of the bar.)

existence enforces localized displacements, excepting the $n = 1$ node; in particular the conclusion can be drawn that such shells do not have pronounced resonances for $n \neq 1$.

No such conclusion can be drawn for $n = 1$, as the infinitely long shell for low frequencies can not be used as approximation for a shell of finite length.

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APPENDIX A

Evaluation of the Integral in Eq. (14) for Large Values of r

The integral in Eq. (14) may be considered in the complex plane and can be evaluated along a suitable selected alternative path. When distorting the path one must consider that the restrictions in Eq. (9) on β impose a restriction on ξ, and it is easily seen that $\xi = |\xi| e^{i\theta}$ is only permitted to assume values such that $0 \leq \theta \leq \pi/2$ or $\pi \leq \theta \leq 3\pi/2$.

The original path of integration as shown in Fig. 2 is interrupted at each pole ξ_j. This path is altered in two ways, Fig. A-1:

1. It is made continuous by adding semi-circular paths of radius ϵ above the real axis for positive poles and below for negative ones. This addition increases the value of the integral by

$$\sum_j \pm \pi i R_j$$

where R_j is the residue of the integral at ξ_j .

2. The path near the branch points is distorted to a semi-circle of radius $\bar{\epsilon}$. As the path remains on the proper side of the branch cut the value of the integral does not change; if a pole occurs within this circle its residue is already accounted under 1.

The value J of the integral in Eq. (14) is

$$J = \int \frac{H_n^{(2)}(\beta r)\, G(\xi)\, e^{i\xi z}\, d\xi}{H_n^{(2)}(\beta a)\left[1 - \frac{2m}{aN\Delta(a\beta)}\right]} + \sum_j \pm \pi i R_j \tag{A-1}$$

where the sign under the sum is the same as that of the respective value ξ_j .

The use of a complex path introduces a difficulty concerning the function N defined by Eq. (6). This equation, taken from Ref. 1, applies only for real values of ξ, because it refers to the frequencies ω_k of modes of the cylindrical shell in vacuo having a half wave length $L = |\pi/\xi|$. If ξ becomes complex this definition loses its meaning. However, one can visualize a solution of the problem under consideration by integral transform, in which case the solution would contain an integral similar to Eq. (A-1). Such a solution would give a more general definition of the function N valid also for complex values of ξ. It is therefore assumed that N may be considered as a function of the complex variable ξ, although only the values for real ξ can be computed from Eq. (16). It will be seen later that values of N for complex argument will not be required for the present purposes.

Consider the values of

$$\beta = \sqrt{\frac{\Omega^2}{c^2} - \xi^2}$$

along the new path. The smallest values of $|\beta|$ occur when ξ goes through the semi-circles in the vicinity of $\xi = \pm \Omega/c$. For small values of $\bar{\epsilon}$ the absolute value $|\beta|$ has a lower bound

$$\min |\beta| > \sqrt{2 \frac{\Omega}{c} \bar{\epsilon}} \tag{A-2}$$

This being the case, one can select a radius R large enough so that for all values of $r \gg R$ the asymptotic expression

$$\lim_{r \to \infty} H_n^{(2)}(\beta r) = e^{i\pi \frac{2n+1}{4}} \frac{e^{-i\beta r}}{\sqrt{\frac{1}{2}\pi\beta r}} \tag{A-3}$$

may be used. Substituting into Eq. (A-1)

$$\lim_{r \to \infty} J = \sqrt{\frac{2}{\pi}}\, e^{i\pi \frac{2n+1}{4}} \int e^{i(\xi z - \beta r)} \frac{G(\xi) d\xi}{\sqrt{\beta r}\, H_n^{(2)}(\beta a) \left[1 - \frac{2m}{\rho a N \Delta(a\beta)}\right]}$$

$$+ \sum_j \pm \pi i \lim_{r \to \infty} R_j \quad . \; . \; . \; . \tag{A-4}$$

This integral is of the form

$$I = \int_A e^{rf(\xi)}\, \bar{G}(\xi) d\xi \tag{A-5}$$

where $\xi = x + iy$ is a complex variable, and

$$\bar{G}(\xi) = \frac{G(\xi)}{\sqrt{\beta r}\, H_n^{(2)}(\beta a) \left[1 - \frac{2m}{\rho a N \Delta(a\beta)}\right]} \tag{A-6}$$

$$f(\xi) = i\gamma\xi - i\beta = i\left[\gamma\xi - \sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right] \tag{A-7}$$

$$\gamma = \frac{z}{r} \tag{A-8}$$

Integrals of the type (A-5) can be evaluated approximately for large values of r by utilizing the fact that the significant contribution to the value of the integral is due to

the portion of the path in the vicinity of "saddle points" ξ_0 (that is points where $f'(\xi_0) = 0$), provided $\bar{G}(\xi)$ has no singularity on the path.

The condition

$$f'(\xi_0) = i\left[\gamma + \xi_0\left(\frac{\Omega^2}{c^2} - \xi_0^2\right)^{1/2}\right] = 0 \tag{A-9}$$

furnishes two roots

$$\xi_0 = \pm \frac{\Omega}{c} \frac{\gamma}{\sqrt{1+\gamma^2}},$$

of which only one pertains to the branch of the function $f(\xi)$ applicable. Eq. (A-9) may be written

$$\gamma + \frac{\xi_0}{\beta} = 0 \tag{A-9a}$$

and because of the restrictions in Eq. (9), β must be real and not negative if ξ is real. Assuming $z > 0$, $\gamma > 0$ Eq. (A-9a) excludes positive values of ξ_0, and the only saddle point is at

$$\xi_0 = x_0 + iy_0, \quad x_0 = -\frac{\Omega}{c}\frac{\gamma}{\sqrt{1+\gamma^2}}, \quad y_0 = 0 \tag{A-10}$$

The selected path goes through this saddle point and follows in its vicinity the real axis. The function $f(\xi) = \phi + i\psi$ is purely imaginary for real values $\xi = x$ of the argument. The path being such that the real part $\phi = 0$, the situation is suitable for the application of the method of stationary phase, (Ref. 3, p. 505, 506). For large values of r the integral I can be approximated (Ref. 3, Eq. (9))

$$\lim_{r\to\infty} I = \sqrt{\frac{2\pi}{r\left|\psi''(\xi_0)\right|}}\ \bar{G}(\xi_0)\exp\left[ir\psi(\xi_0) + \frac{\pi i}{4}\right] + O\left(\frac{1}{r}\right) \tag{A-11}$$

where $\bar{G}$ and ξ_0 are defined by Eqs. (A-6) and (A-10), while

$$\psi(\xi_0) = \gamma\xi_0 - \beta_0 \tag{A-12}$$

$$\psi''(\xi_0) = \frac{\Omega^2}{c^2\beta_0^3} \tag{A-13}$$

$$\beta_0 = \sqrt{\frac{\Omega^2}{c^2} - \xi_0^2} \tag{A-14}$$

The square root in Eq. (A-14) must be taken as positive.

Eq. (A-4) becomes

$$\lim_{r \to \infty} J = \frac{2G(\xi_0) \exp\left[i\left(\frac{\pi(n+1)}{2} - \frac{\Omega\sqrt{r^2+z^2}}{c}\right)\right]}{\sqrt{r^2+z^2}\, H_n^{(2)}(a\beta_0)\left[1 - \frac{2m}{\rho a N(L_0)\Delta(a\beta_0)}\right]} + \sum_j \pm \pi i \lim_{r \to \infty} R_j \qquad \text{(A-15)}$$

where the term containing O(1/r) is omitted. The residual terms are still to be determined.

The term $H_n^{(2)}(\beta r)$ in Eq. (14) governs the behavior of the residues R_j. As was stated previously poles occur only for $|\xi_j| > \Omega/c$. We consider first the case

$$|\xi_j| > \frac{\Omega}{c} \qquad \text{(A-16)}$$

as the case $|\xi_j| = \Omega/c$ requires special treatment. Eq. (9) indicates that $\beta = -(iy)$ is a negative imaginary number; the function $H_n^{(2)}(-iyr)$ decays as $e^{-yr}/\sqrt{r}$, which is faster than the first term of Eq. (A-15), the asymptotic behavior of which depends on the term

$$\sqrt{r^2+z^2}$$

in the denominator. Residues for poles $|\xi_j| > \Omega/c$ give therefore no contribution.

If $|\xi_j| = \Omega/c$ the coefficient $\beta = 0$, and the behavior of the residues is governed by the value of

$$\lim_{\beta \to 0} \frac{H_n^{(2)}(\beta r)}{H_n^{(2)}(\beta a)} = \left(\frac{r}{a}\right)^{-n} \qquad \text{(A-17)}$$

If $n \geq 2$ the residues decay faster than the first term of Eq. (A-15), but the cases $n = 0$ and $n = 1$ still remain.

Even the cases $n = 0$ and 1 do not give contributions because the values R_j of the residues vanish if the poles are at $\pm \pi/c$. This fact can be shown by a routine computation not presented here, and is due to the behavior of $H_0^{(2)}$ and $H_1^{(2)}$ and their derivatives for vanishing argument. The residues never give a significant contribution and the first term of Eq. (A-15) represents the entire result.

To obtain an estimate concerning the values of r for which the result is applicable one must consider the two approximations made. The principal approximation lies in Eq. (A-11) which indicates the second neglected term is of the order $\sqrt{a/r}$ smaller than the first. This means $r/a = 50$ will give some measure of approximation. The second approximation is that Eq. (A-3) is usable* only if $\beta r > 2n + 1$. Enlarging the radius

*Eq. (A-3) gives for $n \leq 6$ less than 5% error in the absolute values of H(z) if

$\bar{\epsilon}$ of the semi-circle around the branch point in Fig. A-1, one can go only as far as the saddle point, and min β according to Eq. (A-2) must be less than

$$\beta_0 = \sqrt{\frac{\Omega^2}{c^2} - \xi_0^2} = \frac{\Omega r}{c\sqrt{1+\gamma^2}}$$

This gives the condition

$$r\beta_0 = \frac{\Omega r^2}{c\sqrt{r^2+z^2}} > 2n + 1 \qquad (n \leqslant 6) \qquad \text{(A-18)}$$

Comparing the two conditions for $z = 0$ one finds Eq. (A-18) can usually be disregarded, the exception being quite low frequencies Ω. Eq. (A-18) controls only if $a\Omega/c \leqslant 0.02\ (2n+1)$.

(footnote continued)

$|z| \gg 2n + 1$; the error in the phase may be larger but is of no consequence as it does not affect the stationary phase method as such, and we are ultimately interested in $|\bar{p}|$.

APPENDIX B

Estimate of the Radial Displacement w at Resonance, $n \geq 2$

The purpose of this appendix is to demonstrate that at resonance the displacements w may be substantial at much larger distances from the applied load than one would normally expect. It is intended to apply the results to stiffened shells; for such shells the function N is nearly constant* for $|\xi| < n/a$. The following computations are restricted to the case N = constant for $|\xi| < n/a$.

The information desired is of qualitative nature and rather crude approximations were considered permissible.

The integral in Eq. (32) to be determined is

$$\bar{J} = \int_{-\infty}^{\infty} I d\xi = \int_{-\infty}^{\infty} \frac{e^{i\xi z}\, d\xi}{\frac{\rho a}{2m}\Delta - \frac{1 + i\delta}{N}} \tag{B-1}$$

There are no poles on the path of integration; the integrand has branch points at $\pm\,\Omega/c$ and the path passes below the negative but above the positive one.

The plan of action is as follows: in cases of resonance the integrand I becomes very large in the vicinity of $\xi = \pm\,\xi_R$ where ξ_R defines the directions of the largest radiation, see Secs. V and VI. These large values are caused by poles in the vicinity of $\pm\,\xi_R$ Deducting a function

$$f(\xi) = \Sigma \frac{R_j}{\xi - \xi_j} \tag{B-2}$$

from the integrand, Eq. (B-1) becomes

$$\bar{J} = \int_{-\infty}^{\infty} (I - f) d\xi + \int_{-\infty}^{\infty} f d\xi \tag{B-3}$$

where R_j are the values of the residues. The integrand of the first integral will nowhere be large, and the integral will be small compared to the second one. The integrand of Eq. (B-1) being a symmetric function of z, one can restrict the computation to $z > 0$, and the second integral can be evaluated in an obvious manner by the residue method,

$$\bar{J}(|z|) = 2\pi i \Sigma R_j + \int_{-\infty}^{\infty} [I - f] d\xi \tag{B-4}$$

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*If the shell is not stiffened the approximation N = constant can also be made, but is valid only for much lower values of $|\xi|$.

where only residues for $\mathrm{Im}\,\xi_j > 0$ are to be included.

Determination of the Poles and Residues

The poles ξ_j are in the vicinity of $|\xi_R| < n/a$ and it will be assumed that N in this range can be treated as a constant. Expressing the function Δ in terms of J and Y functions (as in Sec. V) the integrand becomes

$$I = -\frac{a\beta(J_n' - iY_n')\,e^{i\xi z}}{\frac{\rho a}{m}(J_n - iY_n) + \frac{1+i\delta}{N}\,a\beta(J_n' - iY_n')} \qquad \text{(B-5)}$$

where

$$a\beta = a\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\ .$$

The Bessel functions J and N have the argument $a\beta_R$ and the primes indicate derivatives with respect to the argument,

$$J' = \frac{dJ}{d(a\beta_R)}$$

The subscripts n will be omitted in the following for simplicity.

At resonance the otherwise dominant imaginary part of the denominator of Eq. (B-5) vanishes for

$$\xi = \pm\xi_R,\ \beta = \beta_R = \sqrt{\frac{\Omega^2}{c^2} - \xi_R^2}\ ;$$

this gives the relation

$$-\frac{\rho a}{m}\,Y(R) - \frac{a\beta_R}{N}\,Y'(R) + \delta\,\cancel{\frac{a\beta_R}{N}}\,J'(R) = 0 \qquad \text{(B-6)}$$

where the notation $Y(R)$ stands for $Y(a\beta_R)$.

It is inherent that in the resonant range Y and $Y' \gg J$ and J' ; J and J' will, therefore, be neglected versus Y and Y', but not versus products δY and $\delta Y'$. Dropping the last term in Eq. (B-6), this equation is used to eliminate N in Eq. (B-5),

$$I = \frac{m}{\rho a}\,\frac{ia\beta Y'Y'(R) - a\cancel{\beta J'Y'(R)}}{D}\,e^{i\xi z} \qquad \text{(B-7)}$$

where

$$D = J\,Y'(R) - Y(R)\,J'\frac{\beta}{\beta_R} - \delta Y(R)\,Y'\frac{\beta}{\beta_R} + i\left[-Y\,Y'(R) + Y(R)\,Y'\frac{\beta}{\beta_R} - \delta\cancel{Y(R)\,J'\frac{\beta}{\beta_R}}\right] \qquad \text{(B-8)}$$

Negligible terms which will not be carried further have been crossed out.

To find the poles, let $D = D(a\beta) = 0$. Knowing that the roots occur near $a\beta_R$ the function D is expanded in a power series in terms of

$$\epsilon = a\beta - a\beta_R \tag{B-8a}$$

giving

$$D = D(R) + \epsilon D'(R) + \ldots = 0 \tag{B-9}$$

where $D(R) = D(a\beta_R)$. Using the first two terms of the series only, and noting the relation for the Wronskian, (see Eq. (25))

$$D = JY' - YJ' - \delta YY' = \frac{2}{\pi a\beta_R} - \delta YY' \tag{B-10}$$

where all functions are of argument $a\beta_R$ (the notation J(R) is no longer necessary). In determining D' it is found that the entire contribution from the real terms is small versus the one from the imaginary ones, and only

$$D' = i\left[-Y'^2 + \frac{1}{a\beta_R} YY' + YY''\right] = i\left[Y'^2 + \left(1 - \frac{n^2}{a^2\beta_R^2}\right)Y^2\right] \tag{B-11}$$

remains. The root ϵ of Eq. (B-9) is

$$\epsilon = -\frac{D}{D'} = -i\,\frac{\frac{2}{\pi a\beta_R} - \delta YY'}{Y'^2 + \left(1 - \frac{n^2}{a^2\beta_R^2}\right)Y^2} \tag{B-12}$$

It is necessary to point out that the denominator of Eq. (B-12) changes sign as the argument $a\beta_R$ goes from zero to n, and will be 0 somewhere; in the vicinity of this point the neglected terms will dominate. It can be shown that this occurs at a point where J is no longer very much smaller than Y, that is, it occurs outside the resonant range.

Solving Eq.(B-8) for the values of ξ corresponding to the root ϵ gives the location of the poles

$$\xi_j^2 = \xi_R^2 - \frac{\epsilon^2}{a^2} - 2\beta_R\,\frac{\epsilon}{a} \sim \xi_R^2 - \frac{2\Omega}{ac}\epsilon \quad [j = 1, 2] \tag{B-13}$$

Only the pole ξ_1 which lies near $-\xi_R$ has a positive imaginary part and is to be included in Eq. (B-4). The required residue R_1 of the integrand I, Eq. (B-7), at $\xi = \xi_1$ is

$$R_1 = \frac{m}{\rho a}\,\frac{ia\beta Y\,Y'(R)}{\frac{dD}{d\xi}}\,e^{i\xi z}\Bigg|_{\xi=\xi_1} = \frac{m}{\rho a}\,\frac{ia\beta Y\,Y'(R)}{D'\,\frac{d(a\beta)}{d\xi}}\,e^{i\xi z}\Bigg|_{\xi=\xi_1} \tag{B-14}$$

Noting that ϵ is small the argument $a\beta$ of the various functions can approximately be taken as $a\beta_R$; D' is then given by Eq. (B-11); using

$$\left.\frac{d(a\beta)}{d\xi}\right|_{\xi=\xi_1} = -\frac{a\xi_1}{\beta_R} \qquad \text{(B-14)}$$

one obtains

$$R_1 = Ae^{i\xi_1 z} \qquad \text{(B-15)}$$

where

$$A = -\frac{m}{\rho a}\,\frac{\beta_R^2 Y_n Y_n'}{\xi_1\left[Y_n'^2 + \left(1 - \frac{n^2}{a^2\beta_R^2}\right) Y_n^2\right]} \qquad \text{(B-16)}$$

Estimate of the Remainder Integral in Eq. (B-4)

It is assumed that the damping term in Eq. (B-1) can be neglected. Asymptotic values for large ξ will be required for $f(\xi)$ and N. Noting $\xi_2 = -\xi_1$ and $R_2 = -Ae^{-i\xi_1 z}$ one obtains from Eqs. (B-2, 15, 16)

$$\lim_{\xi\to\infty} f(\xi) = \frac{2\xi_1 \cos \xi_1 z}{\xi^2} A + \frac{2i \sin \xi_1 z}{\xi} A \qquad \text{(B-17)}$$

The asymptotic value of N is obtained from the following consideration: for $\xi \to \infty$ the radial, tangential and longitudinal vibrations (or waves) in a thin shell will be essentially like plane waves in a flat plate, and will not be coupled. The values α_k, Eq. (7); depend on the radial component of the vibration; due to the uncoupling $\alpha_1 = 1$ for the radial (i.e. bending) vibrations, and $\alpha_2 = \alpha_3 = 0$ for the others. The bending frequency can be written $\omega_1 = \bar{c}_s\xi$ where $\bar{c}_s$ is the velocity of propogation of these waves, i.e. the velocity of shear waves in the shell material. Eq.(B-6) gives

$$\lim_{\xi\to\infty} N \to \frac{\alpha_1\Omega^2}{\omega_1^2 - \Omega^2} \to \frac{\Omega^2}{\bar{c}_s^2\,\xi^2} \qquad \text{(B-18)}$$

where Ω^2 was neglected versus ω_1^2.

In the beginning of the appendix it was assumed that N is constant for small ξ; the value of N is implied by Eq. (B-6)

$$N = -\frac{m}{\rho a}\,\frac{a\beta_R Y_n'(R)}{Y_n(R)} \simeq \frac{mn}{\rho a} \qquad \text{(B-19)}$$

the approximation being permissible as the argument of the Bessel function is less than n, (see Eq. (B-30)). There is necessarily a transition between the extreme values given by Eq. (B-18, 19); however, as an approximation it will be assumed here that these relations are valid up to the intersection of the two curves at a value $\xi = \pm \xi_L$,

$$\xi_L = \frac{\Omega}{\bar{c}_s} \sqrt{\frac{\rho a}{mn}} \tag{B-20}$$

The remainder integral may be written, noting $I(\xi) = I(-\xi)$

$$\int_{-\infty}^{\infty} (I - f)d\xi = - \int_{-\infty}^{-\xi_L} fd\xi - \int_{\xi_L}^{\infty} fd\xi + \int_{-\xi_L}^{\xi_L} (I - f)d\xi + 2 \int_{\xi_L}^{\infty} Id\xi \tag{B-21}$$

It is now reasoned that near the origin, i.e. $|\xi| < \xi_L$ the integrand $(I - f)$ must be of the order of magnitude of N, as the extremely high values which occured due to the cancellation of the Δ and N terms, have been removed. The third integral is therefore, of the order

$$\int_{-\xi_L}^{\xi_L} [I - f]d\xi \simeq 2N\xi_L = 2 \frac{\Omega}{\bar{c}_s} \sqrt{\frac{mn}{\rho a}} \tag{B-22}$$

The first and second integral can be found by substituting Eq. (B-17)

$$\int_{-\infty}^{-\xi_L} fd\xi + \int_{\xi_L}^{\infty} fd\xi = 4 \frac{\xi_1 \bar{c}_s}{\Omega} \sqrt{\frac{mn}{\rho a}} A \cos \xi_1 z \tag{B-23}$$

The fourth integral will have its largest value for $z = 0$, giving the limit

$$2 \int_{\xi_L}^{\infty} Id\xi = - \frac{2\Omega^2}{\bar{c}_s^2} \int_{\xi_L}^{\infty} \frac{e^{i\xi z}}{\xi^2} d\xi \leq \frac{2\Omega}{\bar{c}_s} \sqrt{\frac{mn}{\rho a}} \tag{B-24}$$

Collecting the various terms the value of J becomes

$$J(z) \sim 2\pi i A \, e^{i\xi_1 z} - 4A \frac{\xi_1 \bar{c}_s}{\Omega} \sqrt{\frac{mn}{\rho a}} \cos \xi_1 z \tag{B-25}$$

valid if the first term is dominant, and provided that the terms originating from Eqs. (B-22, 24) are small

$$2 \frac{\Omega}{\bar{c}_s} \sqrt{\frac{mn}{\rho a}} << A \tag{B-26}$$

It is intended to apply this result only in cases where ξ_I is very small; both terms of Eq. (B-25) represent in such cases functions which are nearly constant for $|z| \lessdot a$, and it is this fact which is the point at issue. For the present purpose the second term of Eq. (B-25) which does not change the character of the solution can therefore be neglected, even if it is much larger than the terms excluded by Eq. (B-26). The value of $\bar{J}$ becomes finally

$$\bar{J}(z) \sim 2\pi i A\ e^{i\xi_I z} \tag{B-27}$$

where the condition (B-26) is vital, while it is sufficient to require less stringently

$$a\left|\xi_I\right| < \frac{\pi}{2}\ \frac{a\Omega}{\bar{c}_s} \sqrt{\frac{\rho a}{mn}} \tag{B-28}$$

Simplifications for Low Frequencies

Useful simplifications are possible if the frequency is low $a\Omega/c \lessdot n/2$; these approximations cover the most interesting range of high radiation, but are not suitable for the transition range of gradually diminishing radiation as $a\Omega/c$ increases toward n.

The dominant term in the expansions of Y_n for small arguments is

$$Y_n(x) \sim -\frac{2^n(n-1)!}{\pi x^n} \tag{B-29}$$

giving

$$Y_n' = -\frac{n}{x} Y_n + Y_{n-1} \sim -\frac{n}{x} Y_n \tag{B-30}$$

$$Y_n'^2 + \left(1 - \frac{n^2}{x^2}\right) Y_n^2 = Y_n^2 + Y_{n-1}^2 - \frac{2n}{x} Y_n Y_{n-1} \sim -\frac{1}{n-1} Y_n^2 \tag{B-31}$$

Substituting these relations into Eq. (B-12)

$$\epsilon = i\left[\frac{2(n-1)}{\pi a\beta_R} Y_n^{-2} + \frac{n(n-1)\delta}{a\beta_R}\right] \tag{B-32}$$

where $Y_n = Y_n(a\beta_R)$. Substitution of this expression in Eq. (B-13) gives ξ_I; in the case $\xi_R = 0$, the term $a\beta_R$ becomes $a\Omega/c$, and one obtains

$$a\xi_I = e^{3i\frac{\pi}{4}} \sqrt{\frac{4(n-1)}{\pi} Y_n^{-2} \left(\frac{a\Omega}{c}\right) + 2n(n-1)\delta} \qquad (\xi_R \simeq 0) \tag{B-33}$$

Eq. (B-16) gives

$$A = -\frac{n(n-1)m}{\rho a} \frac{\beta_R}{a\xi_I} \tag{B-34}$$

The vital condition (B-26) for the validity of above results becomes

$$|a\xi_I| \ll (n - 1)\sqrt{\frac{nm}{\rho a}}\,\frac{\bar{c}_s\beta_R}{2\Omega} \tag{B-35}$$

If $\xi_R = 0$ or $\xi_R \sim 0$, then $\beta_R = \Omega/c$ and

$$|a\xi_I| \ll (n - 1)\sqrt{\frac{nm}{\rho a}}\,\frac{\bar{c}_s}{2c} \qquad (\xi_R \simeq 0) \tag{B-35a}$$

This condition now shows clearly that the results of this appendix are valid for small ξ_I where they are to be applied. The second condition (B-28) does not simplify for small arguments.

<u>Conclusion</u>:

Substituting Eq. (B-27) into Eq. (32) and noting that A is not a function of z, the displacement can be expressed in terms of the displacement w(0) at z = 0,

$$w(z) = w(0)e^{i\xi_I z} \tag{B-36}$$

where the complex quantity ξ_I follows from Eqs. (B-12, 13). The relation (B-36) is valid only if the condition (B-26) is strictly satisfied, and Eq. (B-28) is at least not badly violated. For small arguments Eqs. (B-32, 33 and 35) can be used instead of Eqs. (B-12, 13 and 26), respectively.

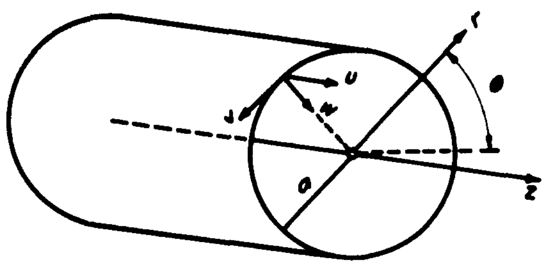


FIG. 1

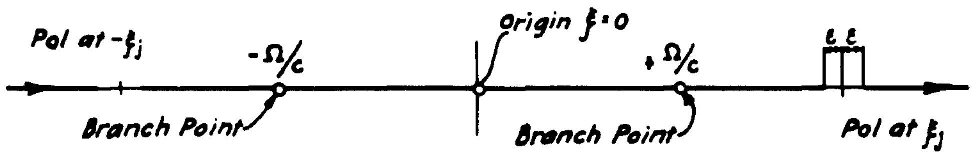


FIG. 2. Path of Integration in ξ - Plane

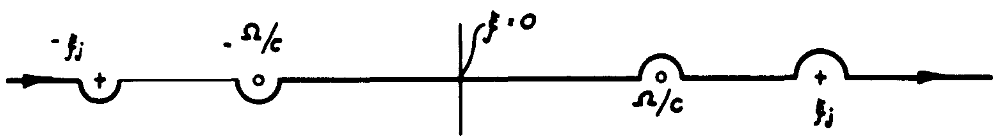


FIG. 2a. Modified Path of Integration

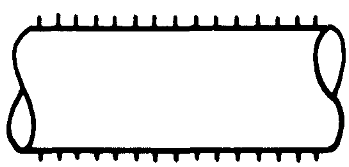


FIG. 3

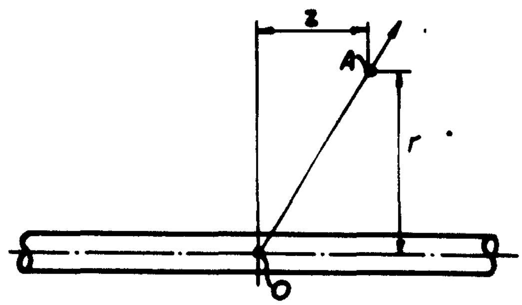


FIG. 4

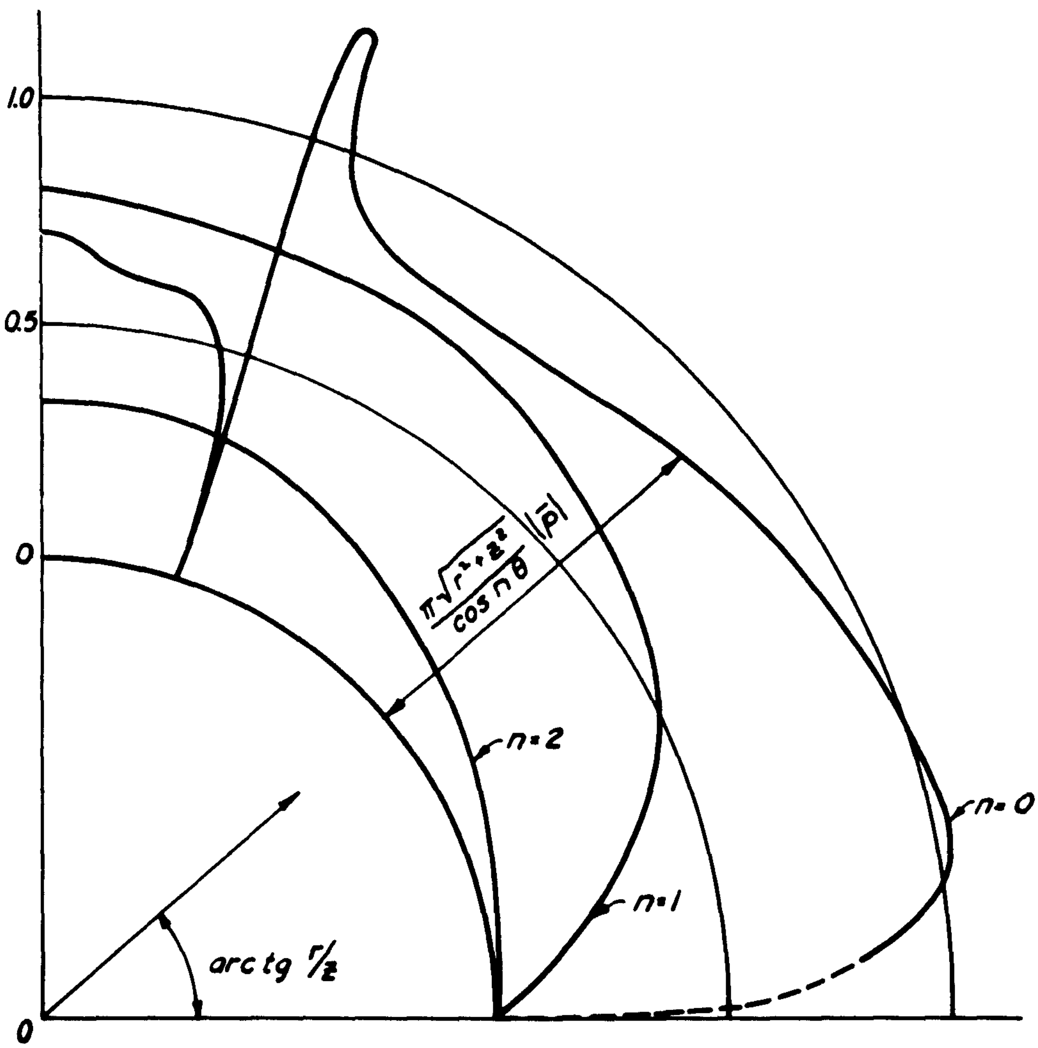


Fig. 5 Pressure $|\bar{p}|$ as Function of r/z

$\frac{\rho a}{2m} = 4$, $a\Omega/c = 0.8$

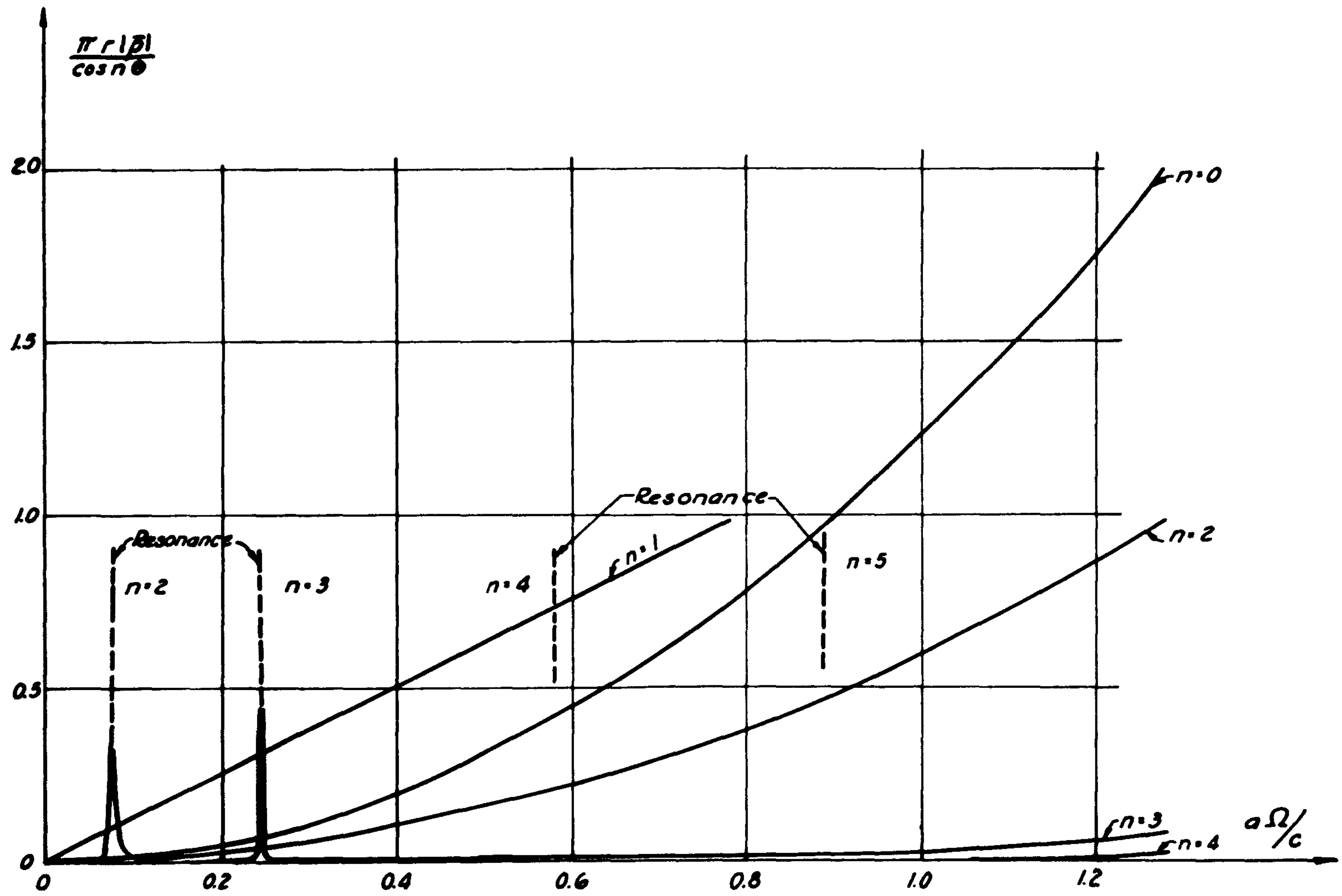


Fig. 6a. Pressure $|\beta|$ in the Plane $z = 0$ as Function of $a\Omega/c$

$\frac{\rho a}{2m} = 4$, $n = 0$ to 4

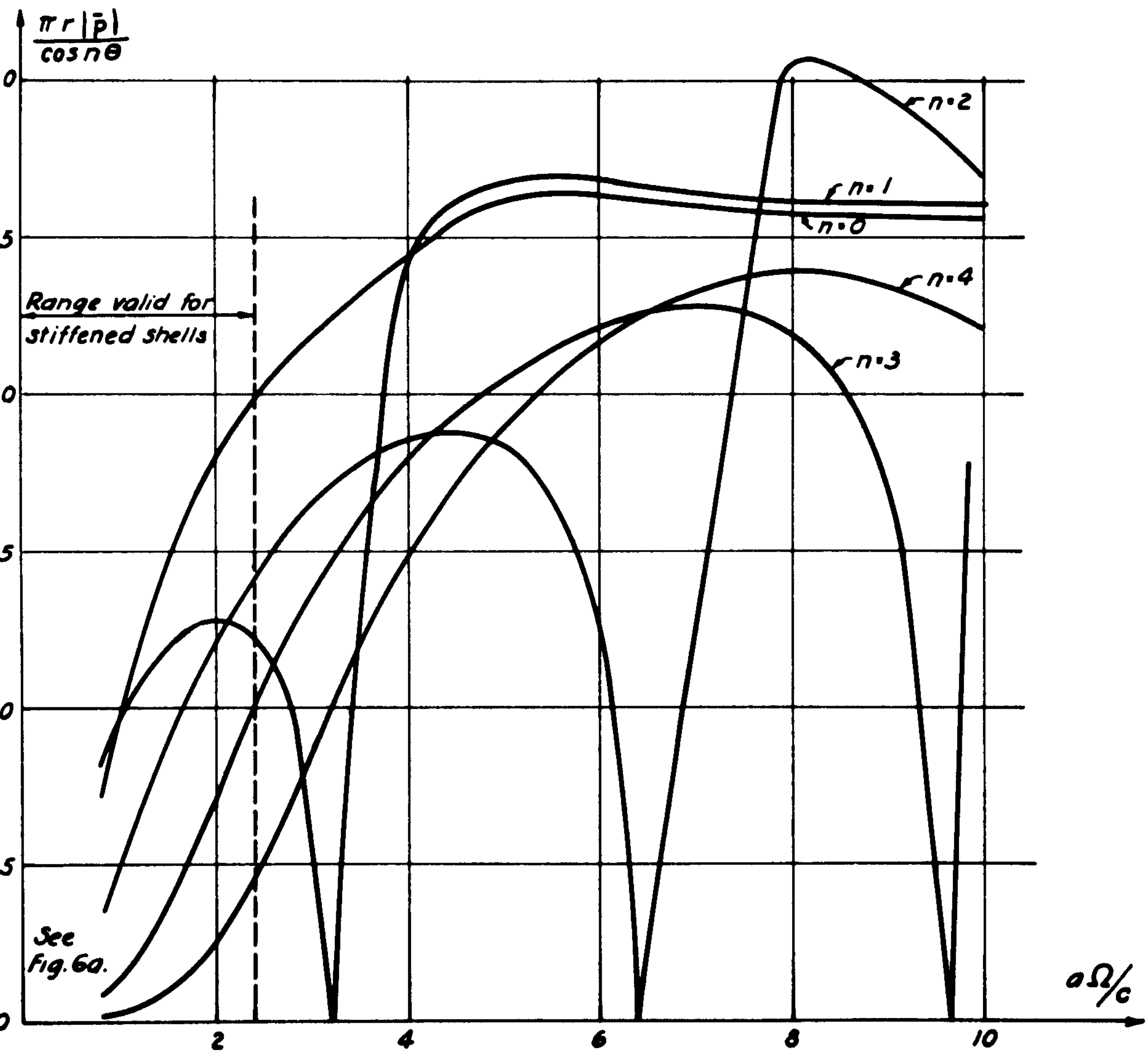


Fig. 6b. Pressure $|\bar{p}|$ in the Plane $z=0$ as Function of $a\Omega/c$

$\frac{\rho a}{2m} = 4 \quad n = 0$ to 4

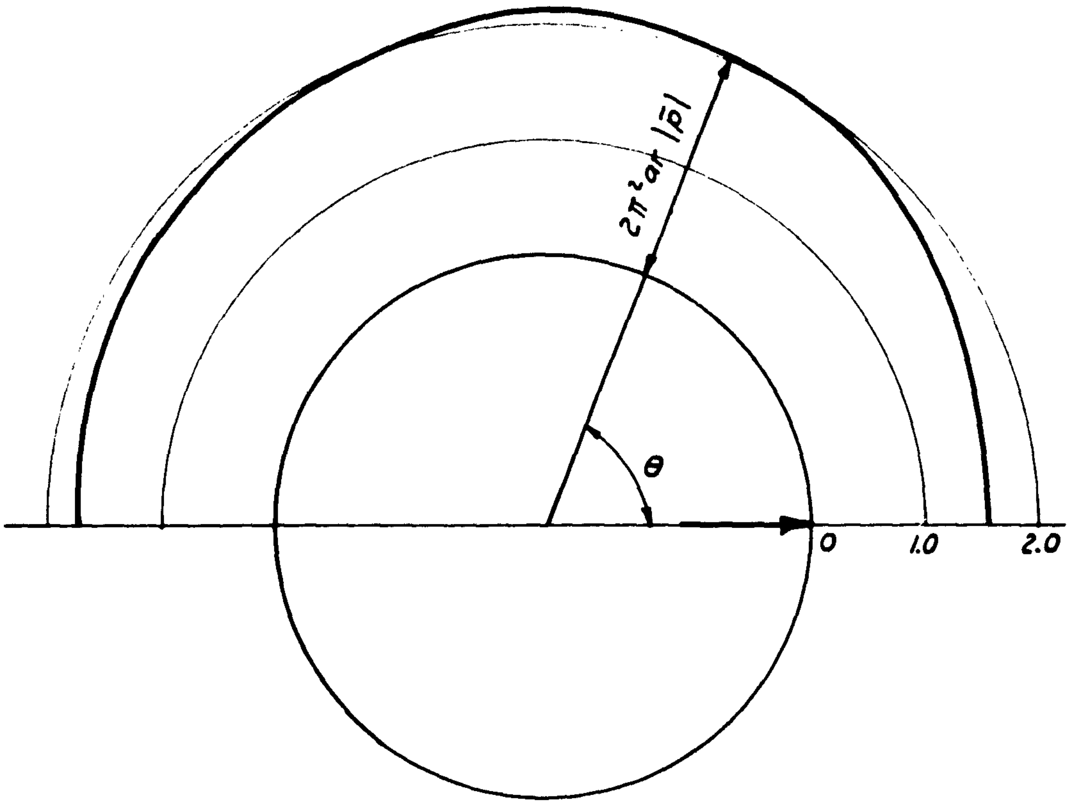


FIG. 7 Pressure $|\bar{p}|$ in the Plane $z=0$ due to Concentrated Load at $z=\theta=0$

$\frac{\rho a}{2m} = 4 \quad a\Omega/c = 0.80$

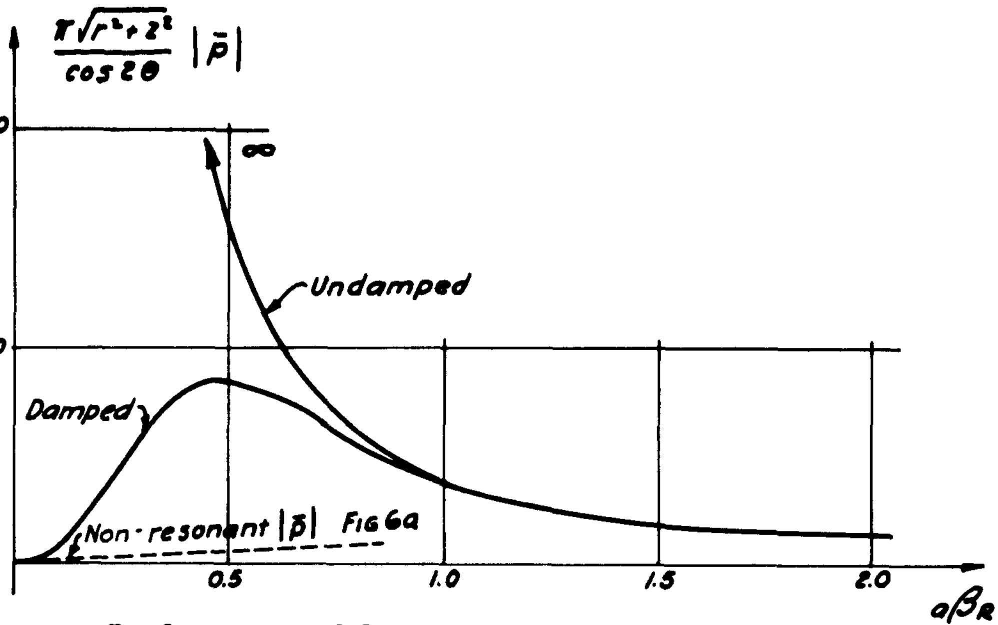


Fig. 8a. Level of Pressure at Resonance, $n = 2$, $\delta = 0.01$

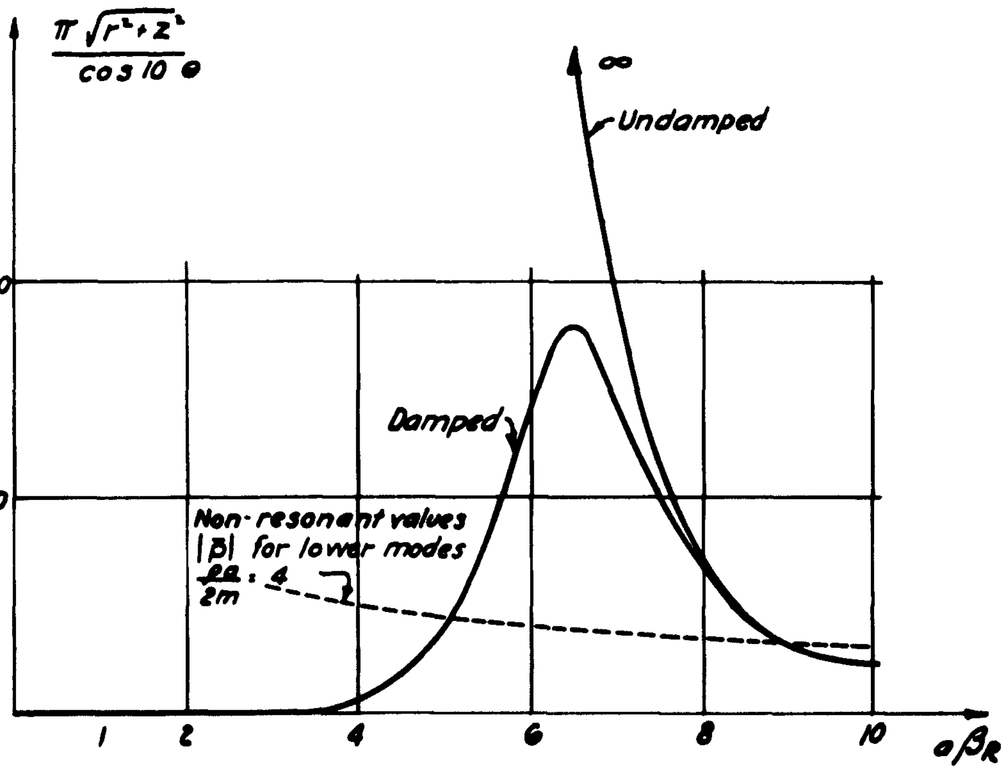
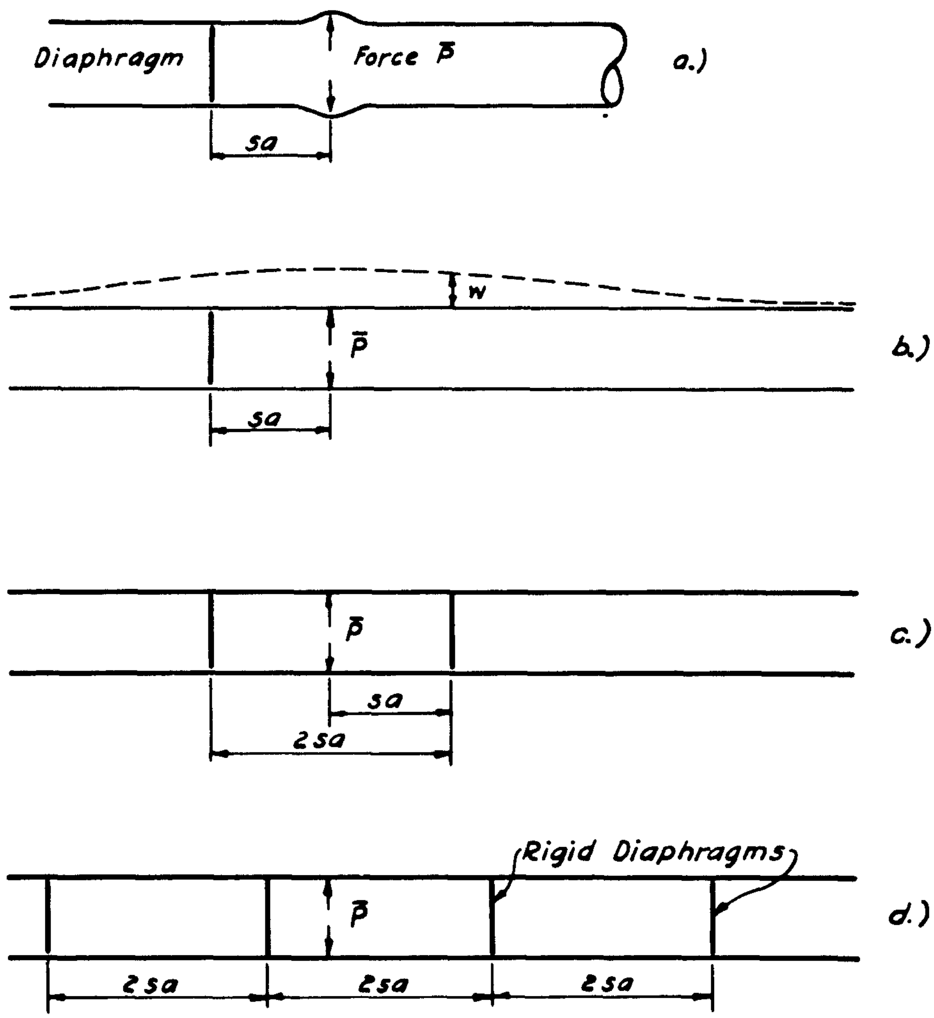
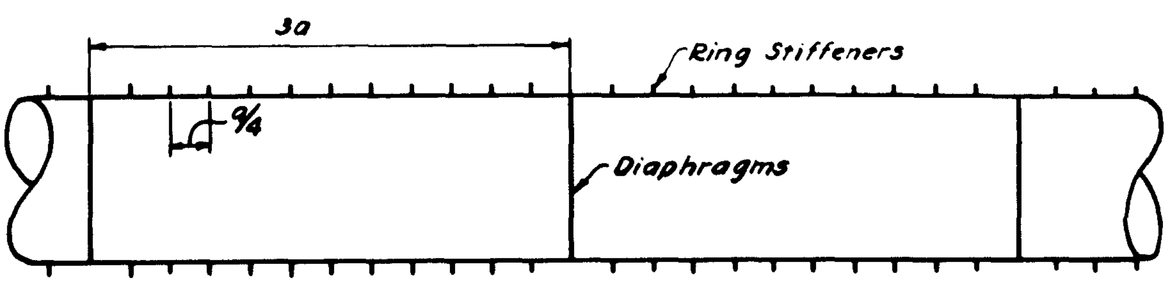


Fig. 8b. Level of Pressure at Resonance, $n = 10$ $\delta = 0.01$



Diaphragm
Force $\bar{P}$
a.)
sa
w
$\bar{P}$
b.)
sa
$\bar{P}$
c.)
sa
$2sa$
Rigid Diaphragms
$\bar{P}$
d.)
$2sa$
$2sa$
$2sa$

FIG. 9



$3a$
Ring Stiffeners
$a/4$
Diaphragms

FIG. 10

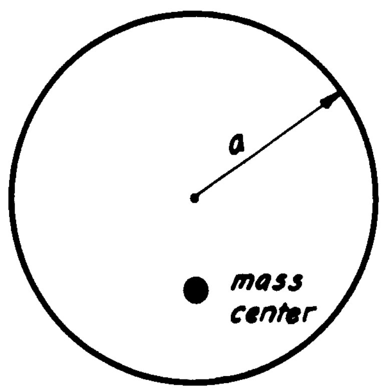


Fig. 11

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